The Impact of Vertical Integration on
Losses from Collusion

by

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Abstract

Upstream collusion that increases the price of an input can harm an independent downstream supplier (D). We ask whether this harm is more or less pronounced when D's downstream rival is a vertically integrated supplier. The answer depends on the nature and intensity of downstream competition. Vertical integration (VI) tends to increase D's loss from collusion when downstream competition is relatively intense or D is a relatively strong competitor. VI tends to reduce D's loss from collusion when downstream competition is relatively limited and D is a relatively weak competitor or when the market demand for the downstream product is highly price inelastic.

Key Words: collusion, vertical integration, antitrust damages.

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1 Introduction

An extensive literature analyzes the effects of vertical integration on industry competition and economic welfare.[1] The literature considers such issues as whether a vertically-integrated firm might decline to supply a critical input to downstream rivals,[2] how vertical integration can affect the intensity of downstream competition,[3] and when vertical integration can enhance the ability of an upstream supplier to commit to charge a relatively high input price.[4] The literature also devotes considerable attention to determining whether vertical integration facilitates or impedes collusion by upstream suppliers.[5]

We adopt a different focus. We take as given the ability of upstream suppliers to collude in setting the terms on which they sell a key input to downstream suppliers,[6] and examine how vertical integration affects the damage the collusion imposes on unaffiliated downstream suppliers.[7] Collusive agreements among ostensible competitors are illegal in most jurisdictions.[8] Whereas the welfare loss caused by collusive agreements provides a rationale for public enforcement of the antitrust laws, the conversion of consumer surplus into cartel profit provides the economic rationale for private enforcement. In the United States, Section 4 of the Clayton Act (15 U.S.C. §15) provides that "[a]ny person who shall be injured in his business or property by reason of anything forbidden in the antitrust laws may sue therefor. . . and shall recover threefold the damages by him sustained . . . ".

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[1]Informative reviews of this literature include Riordan and Salop (1995), Rey and Tirole (2007), and Riordan (2008).

[2]See Salinger (1988), Hart and Tirole (1990), Ordover et al. (1990), Chen and Riordan (2007), Rey and Tirole (2007), Levy et al. (2018), and Nocke and Rey (2018), for example.

[3]See Chen (2001), for example.

[4]See, for example, Reiffen (1992), McAfee and Schwartz (1994), and Nocke and Rey (2018).

[5]See, for example, Nocke and White (2007), Riordan (2008), Mendi (2009), Normann (2009), Nishiwaki (2016), and Biancini and Ettinger (2017).

[6]Symeonidis (2004), among others, analyzes the effects of collusion among *downstream* suppliers. He identifies conditions under which such collusion can increase welfare in a setting where upstream and downstream suppliers bargain over the price of an input.

[7]An unaffiliated supplier is a supplier that is not vertically-integrated with any other entity.

[8]In the United States, for example, §1 of the Sherman Antitrust Act of 1890 forbids "every contract, combination. . . , or conspiracy in restraint of trade. . . " (150 U.S.C. §1).

In practice, it is often difficult to estimate precisely the damage caused by collusion, in part because the magnitude of the damage can vary with diverse features of the prevailing environment. Scholars (e.g., Verboven and van Dijk, 2009) have observed that vertical integration can affect the damages from collusion, in part by affecting the profit-maximizing behavior of industry participants. However, the literature does not provide general guidance as to when vertical integration is likely to systematically increase or reduce damages from collusion. The objective of this research is to begin to provide this guidance.[9]

We develop this initial guidance in a setting where upstream producers can collude both in setting the price at which an essential input is sold and in structuring associated supply relations. Specifically, the upstream producers can specify the fraction of a downstream producer's demand for the input that each of them will supply.[10] Although such structuring of supply relationships has been documented in practice (e.g., Marshall and Marx, 2014, p. 120), it is seldom analyzed in detail in formal models of collusion.[11,12] We find that such structuring can affect not only the quantitative but also the qualitative effects of vertical integration on the damages from collusion. Indeed, in the presence of vertical integration, upstream collusion can sometimes increase the profit of an unaffiliated downstream producer.

Collusion among vertically-integrated suppliers has been alleged or documented in many

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[9] We examine the impact of upstream collusion on the profit of a downstream supplier. We do not analyze the corresponding social losses from upstream collusion that other studies (e.g., Basso and Ross, 2010; Boone and Müller, 2012) examine in distinct settings.

[10] Utton (2011, p. 6) observes that a focus on a collusive price is "a useful first approximation," but notes that collusion also typically entails an agreement on how the total demand induced by the collusive price is allocated among industry suppliers. Utton further notes (p. 9) how a central sales agent might sometimes help to implement the agreement. In addition, Utton (pp. 47-48) describes some of the means by which vitamin suppliers enforced collusive output allocation agreements in the 1990s. The practices included refusals by some suppliers to fill orders placed by certain potential customers. Scherer (1980, pp. 193-197) explains how strategic inventory management might be employed to implement collusive price and sales agreements.

[11] Piccolo and Miklos-Thai (2012) consider the management of supply relations, but examine how supply contracts can be structured to enhance the ability of *downstream* suppliers to collude.

[12] Harrington (2017, p. 31) observes that, in practice, colluding suppliers often set a common sales price and assign potentially asymmetric market shares. He further observes that the literature typically considers settings with symmetric firms, and so focuses on the collusive price-setting, abstracting from the collusive management of market supply. Our model allows the downstream suppliers to be asymmetric, and we examine how the relevant asymmetries affect the optimal structuring of supply relations.

court cases. To illustrate, some forty years ago, the nation's sugar refiners colluded to set the price at which they sold refined sugar to candy manufacturers.[13] Some of the candy manufacturers were vertically-integrated with sugar refiners (e.g., subsidiaries of Borden) whereas other manufacturers were unaffiliated producers. More recently, suppliers of dynamic random access memory (DRAM) were found to have colluded in setting the terms on which they sold DRAM to manufacturers of personal computers.[14] Some of the DRAM suppliers were vertically-integrated entities that also sold personal computers (e.g., Samsung and Micron Technology). As one further example, suppliers of lithium ion batteries were alleged to have collusively set the terms on which they sold batteries to manufacturers of assorted electronics products, including telephones and laptop computers.[15] Some, but not all, of the battery suppliers (e.g., Sony and Toshiba) also manufactured these electronics products.[16]

Our formal model considers the interaction between two upstream suppliers (U1 and U2) and two downstream suppliers (D1 and D2), allowing for the possibility that U1 and D1 might be vertically integrated. We find that such vertical integration tends to increase the loss that upstream collusion between U1 and U2 imposes on the unaffiliated downstream supplier (D2) in the presence of relatively intense downstream competition or when D2 is a relatively strong competitor. In contrast, vertical integration can reduce D2's loss from collusion when D2 is a relatively weak competitor and downstream competition is limited or when the market demand for the downstream product is highly price inelastic.

We derive these conclusions by examining three forms of downstream competition. Un-

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[13] *In re Sugar Industry Antitrust Litigation*, 579 F. 2d 13 (3rd Cir. 1978).

[14] *In re Dynamic Random Access Memory (DRAM) Antitrust Litigation*, (N.D. Cal,. No. 02-01486, 06/27/14).

[15] *In re Lithium Ion Batteries Antitrust Litigation*, (N.D. Cal., No. 4:13-md-02420-YGR, 3/6/16).

[16] Suppliers of such products as LCD screens (*In re TFT-LCD Antitrust Litigation*, (N.D. Cal., No. 3:07-md-01827, 4/20/07)), optical disc drives (*In re Optical Disk Drive Antitrust Litigation*, (N.D., No. 3:10-md-02143-RS, 10/3/14)), electronic capacitors (*In re Capacitors Antitrust Litigation*, (N.D. Cal., No. 3:14-cv-03264-JD, 12/30/15)), and automobile components (e.g., hoses, airbags, and steering wheels) (*U.S. v. Toyoda Gosei Co., Ltd.*, (N.D. Ohio, No. 3:14-cr-00349-JZ, 9/29/14)) have also been accused of collusion. Some of these upstream producers are vertically-integrated suppliers that also manufacture products (e.g., televisions, computer monitors, personal computers, and automobiles) that employ the identified inputs.

der Cournot competition, vertical integration (VI) always increases the loss that upstream collusion imposes on D2. In this setting, the colluding upstream suppliers increase the price of the input above its production cost, and do so to a greater extent under VI than in its absence. This wedge between price and cost imposes a competitive disadvantage on D2 under VI because D1's concern with U1's profit leads D1 to effectively perceive a lower input price than D2 faces under VI. The higher input price and the competitive disadvantage that D2 experiences under VI ensure that its loss from collusion is more pronounced under VI than in its absence.

A very different conclusion arises in the presence of downstream Hotelling competition with full market coverage. When U1 and U2 collude in this setting under VI, they structure supply relations to reduce the intensity of downstream price competition, in part because higher downstream prices do not reduce the demand for the input. Specifically, the vertically-integrated upstream producer (U1) supplies the input to D2 and the unaffiliated upstream producer (U2) supplies the input to the vertically-integrated downstream producer (D1). Under these supply relations, D1 declines to compete aggressively against D2 under VI because intense competition would reduce D2's demand for the input and thereby diminish U1's (profitable) sales of the input. The ensuing limited intensity of downstream price competition under VI benefits D2.

VI can either increase or reduce D2's loss from collusion in the presence of downstream price competition with linear demand and product differentiation. When D2 is a relatively strong competitor, U1 and U2 set a higher input price under VI than in its absence. This is the case because the input price primarily affects D2's operations under VI, and D2's demand for the input does not decline unduly as the input price is increased (to enhance D1's profit) when D2 is a relatively strong competitor. The higher input price and the competitive disadvantage that a relatively strong D2 faces under VI ensure that D2's loss from collusion is more pronounced under VI than in its absence.

However, if D2 is a relatively weak competitor, it will face a lower input price under VI

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than in its absence, as U1 and U2 act to avoid reducing D2's demand for the input unduly. When the products that D1 and D2 sell are highly differentiated, the competitive disadvantage that D2 faces under VI is relatively inconsequential. Consequently, the dominant effect of VI is the lower collusive input price that D2 faces, so VI reduces D2's loss from collusion.[17]

Our analysis seems closely related to the aforementioned work of Verboven and van Dijk (2009). The authors analyze the impact of an increased input price on the profit of downstream suppliers under different forms of retail competition.[18] Their important study focuses on assessing the relative magnitudes of three effects of the price increase: higher input costs, additional revenue from higher equilibrium retail prices, and reduced revenue due to foregone sales caused by higher retail prices.[19] The authors observe that these effects can vary if a downstream producer is vertically integrated. However, their study is not designed to provide systematic guidance on how vertical integration affects losses from collusion. Their study also focuses on small, exogenous changes in input prices and does not permit colluding upstream firms to structure supply relations with downstream producers.

Our analysis proceeds as follows. Section 2 describes the key elements of our model that are common to each of the three settings we analyze. Sections 3, 4, and 5, respectively, then examine the impact of vertical integration on the losses from upstream collusion under downstream Cournot competition, Hotelling competition, and price competition with differentiated products and linear demand. Section 6 provides concluding observations and identifies directions for future research.[20]

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[17]In contrast, if the products of D1 and D2 are sufficiently homogeneous, the competitive disadvantage that D2 faces under VI ensures that it will incur a larger loss from collusion under VI than in its absence even when D2 is a relatively weak competitor.

[18]Verboven and van Dijk (2009) discuss additional effects of the higher input price, but focus on the effect of the higher price on the profit of downstream suppliers.

[19]See Kosicki and Cahill (2006) and Han et al. (2009), for example, for related studies.

[20]The Appendix outlines the proofs of all formal conclusions. Bet et al. (2019) provides detailed proofs.

2 Common Model Elements

We consider settings where two upstream firms (U1 and U2) and two downstream firms (D1 and D2) interact. U1 and U2 each produce a homogeneous input at constant unit cost $c^u > 0$. D1 and D2 employ the input to produce their retail products. One unit of the input is required to produce each unit of the retail product. Di's incremental unit cost of producing its retail product after acquiring the input is $c_i^d \geq 0$ for $i \in \{1, 2\}$.

We analyze two industry structures. Under vertical separation (VS),[21] neither upstream firm has an ownership interest in a downstream firm. Similarly, neither downstream firm has an ownership interest in an upstream firm. Under vertical integration (VI), U2 and D2 continue to have no ownership interest in another firm. However, U1 and D1 are vertically integrated, so each fully values the profit of both entities.

Our primary goal is to compare the extent to which collusion between U1 and U2 reduces D2's profit under VS and under VI. In the absence of collusion, U1 and U2 act independently to maximize their individual profits. U1 sets the price it charges for the input (w_1) and U2 sets the price it charges for the input (w_2) simultaneously and non-cooperatively. Each upstream firm then supplies all of the demand for its input.

When they collude, U1 and U2 act jointly to maximize their combined profit.[22] They do so by setting the unit price (w) they both charge for the input. U1 and U2 also determine the fraction of each downstream firm's demand for the input that will be supplied by each upstream producer. Formally, U1 and U2 set $f_{ij} \in [0, 1]$ for $i, j \in \{1, 2\}$ $(j \neq i)$, which is the fraction of Dj's demand for the input that is supplied by Ui. Uj supplies the remaining fraction $(1 - f_{ij})$ of Dj's demand for the input.

Industry activity proceeds as follows. Input prices are determined first. Supply relations

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[21] We employ the term "vertical separation" to denote the absence of vertical integration.

[22] By assuming that U1 and U2 act to maximize their joint profit without imposing individual participation constraints, we abstract from the underlying bargaining problem that determines how aggregate profit is divided between the suppliers. In essence, U1 and U2 can be viewed as setting their policy instruments to maximize their joint profit and then implementing a lump-sum transfer payment to achieve the desired rent distribution.

(f_{ij}) are also specified when U1 and U2 collude. Next, D1 and D2 set their choice variables (retail prices or retail quantities) simultaneously and non-cooperatively.[23] D1 and D2 then secure the input quantities required to satisfy the equilibrium demand for their retail products.

3   Cournot Competition

We first consider the setting with downstream Cournot competition, where D1 and D2 produce a homogeneous retail product. Consumer demand for the retail product in this setting is given by the inverse demand function $P(Q) = a - bQ$, where $Q \geq 0$ denotes output and $a > 0$ and $b > 0$ are parameters. Total retail output (Q) is the sum of D1's output (q_1) and D2's output (q_2).

To ensure that D1 and D2 both serve customers in equilibrium under VS, Assumptions 1 and 2 are presumed to hold throughout the analysis in this section.[24] The assumptions ensure that consumers value the retail product highly relative to its production cost and that the unit costs of D1 and D2 are not too disparate.

Assumption 1. $a > c^u + \max \left\{ c_1^d, \, c_2^d, \, 2\,c_1^d - c_2^d, \, 2\,c_2^d - c_1^d \right\}.$

Assumption 2. $\left| c_1^d - c_2^d \right| < \left[2 - \sqrt{3} \, \right] \left[a - c^u - \min \left\{ c_1^d, \, c_2^d \right\} \right].$

The equilibrium in this setting is readily characterized under VS when U1 and U2 do not collude. In this case, U1 and U2 compete vigorously to supply the input to D1 and D2. The intense price competition drives the price of the homogeneous input to its common unit cost (c^u). Conceivably, the input price might differ from c^u under VI because D1's perceived cost of the input purchased from U1 can differ from w_1.[25] However, the competition to serve D2

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[23] D1 and D2 each choose the amount of the homogeneous retail product it will supply under Cournot competition. D1 and D2 each determine the unit price it will charge for its product under the two forms of downstream price competition that we analyze.

[24] The key qualitative change that arises when D1 and D2 do not both serve customers in equilibrium is identified below.

[25] For example, when U1 supplies all of the input that D1 demands (so $f_{11} = 1$), D1 perceives its unit cost of the input to be c^u, regardless of the established value of w_1. This is the case because when D1 decides whether to increase q_1, it considers the sum of the resulting incremental downstream profit that it secures

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compels both U1 and U2 to reduce their input price to cost.

Lemma 1. *In the absence of collusion, $w_1 = w_2 = c^u$, $q_1 = \frac{1}{3b}\left[a - c^u - 2c_1^d + c_2^d\right] > 0$, and $q_2 = \frac{1}{3b}\left[a - c^u - 2c_2^d + c_1^d\right] > 0$ in equilibrium both under VS and under VI.[26]*

The fact that downstream output levels are the same under VI and VS in the absence of collusion merits brief explanation. Conceivably, D1's concern with U1's profit under VI could induce D1 to choose a different output than it chooses under VS. However, when competition drives U1's upstream profit margin to 0, D1's choice of q_1 does not affect U1's upstream profit. Consequently, absent collusion, equilibrium retail outputs do not vary with the vertical structure of the industry.

Because input prices and outputs do not vary with the prevailing vertical industry structure in the absence of collusion, neither does D2's downstream profit. This profit is readily calculated, given the findings in Lemma 1.

Conclusion 1. *In the absence of collusion, D2's equilibrium profit under VS (π_2^S) and its profit under VI (π_2^I) are $\pi_2^S = \pi_2^I = \frac{1}{9b}\left[a - c^u - 2c_2^d + c_1^d\right]^2$.*

Next we characterize the outcomes that arise when U1 and U2 collude. When the upstream suppliers act to maximize their joint profit under VS, they choose w to maximize $\left[w - c^u\right]\left[q_1 + q_2\right]$, recognizing that D$i$ chooses q_i to maximize $\left[P(q_1 + q_2) - w - c_i^d\right]q_i$, taking q_j as given (for $i, j \in \{1, 2\}$, $j \neq i$). Lemma 2 records formally the aforementioned fact that under the maintained assumptions, D1 and D2 both serve customers in equilibrium under VS.

Lemma 2. *$q_1 > 0$ and $q_2 > 0$ in equilibrium under VS and collusion.*

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and the associated incremental upstream profit that U1 secures. The (transfer) price at which U1 sells the input to D1 does not affect this sum of downstream and upstream profit.

[26] For expositional ease, Lemma 1 does not state explicitly the prevailing form of downstream competition (i.e., Cournot competition). The same is true of all other lemmas, conclusions, and propositions throughout the ensuing discussion. All formal results pertain to the setting with downstream Cournot competition in Section 3, to the setting with downstream Hotelling competition in Section 4, and to the setting with downstream price competition, differentiated products, and linear demand in Section 5.

When U1 and U2 collude under VI, they choose w, f_{11}, and f_{12} to maximize the sum of upstream profit, $[w - c^u][q_1 + q_2]$, and D1's downstream profit, $[P(q_1 + q_2) - w - c_1^d]q_1$. In doing so, U1 and U2 recognize that each downstream supplier acts to maximize its objective, taking its rival's output as given. D2 sets q_2 to maximize $[P(q_1 + q_2) - w - c_2^d]q_2$. D1 chooses q_1 to maximize $[P(q_1 + q_2) - w - c_1^d]q_1 + [w - c^u][f_{11}q_1 + f_{12}q_2]$. Lemma 3 reports that when U1 and U2 collude under VI, they ensure D2 only supplies the homogeneous retail product to customers in equilibrium if it is the most efficient downstream supplier (i.e., if $c_2^d < c_1^d$). Otherwise, U1's interest in D1's downstream profit leads U1 and U2 to eliminate D2 as an effective supplier of the homogeneous retail product by increasing w to the point where D2 cannot secure positive profit.

Lemma 3. *In equilibrium, under VI and collusion: (i) $q_1 > 0$ and $q_2 > 0$ if $c_2^d < c_1^d$; and (ii) $q_1 > 0$ and $q_2 = 0$ if $c_1^d \leq c_2^d$.*

Conclusion 2 characterizes the supply relations that U1 and U2 establish when they collude under VI. As discussed further below, when the collusive input price (w^{Ic}) exceeds upstream cost (c^u), D1 expands its output more aggressively as f_{11} increases under VI. As f_{11} increases, D1 secures a larger fraction of its input needs from U1. Because D1 values U1's upstream profit when the two suppliers are vertically integrated, D1 effectively perceives the unit price of the input to decline toward c^u as f_{11} increases toward 1.

Conclusion 2. *If $c_1^d > c_2^d$, then under VI and collusion, $f_{11} = \frac{a - c^u - c_1^d}{a - c^u - c_2^d} \in (\frac{1}{2}, 1)$, so $\lim_{c_1^d \to c_2^d} f_{11} = 1$ and f_{11} increases as a increases, c_2^d increases, c_1^d declines, or c^u declines.*

Conclusion 2 reports that when D1 and D2 both serve customers in equilibrium (because $c_2^d < c_1^d$), U1 supplies the majority, but not all, of D1's demand for the input (i.e., $f_{11} \in (\frac{1}{2}, 1)$) under VI and collusion. By setting $f_{11} > \frac{1}{2}$ and $w > c^u$, U1 and U2 motivate D1 to compete relatively aggressively against D2 in order to expand q_1 and thereby increase U1's upstream profit.[27] The output expansion by D1 also helps to increase its profit, which

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[27]The value of f_{12} does not affect D1's choice of q_1 because D1 takes q_2 as given when choosing q_1. The

U1 values. U1 and U2 set f_{11} below 1 to avoid inducing D1 to act too aggressively in its competition with D2. Excessive aggression would reduce upstream profit by reducing unduly the output of the more efficient downstream supplier (D2). The concern with reducing D2's output diminishes, and so U1 and U2 increase f_{11}, as D2's cost advantage over D1 declines (i.e., as c_1^d declines or c_2^d increases).[28]

U1 and U2 also increase f_{11} as a increases, which expands potential downstream profit. By increasing f_{11}, U1 and U2 induce D1 to compete more aggressively and thereby secure a relatively large fraction of the more pronounced downstream industry profit. U1 and U2 also increase f_{11} as c^u declines, which causes upstream profit to increase more rapidly as downstream industry output increases. By increasing f_{11}, U1 and U2 induce D1 to expand its output, which serves to increase total equilibrium industry downstream output.

Lemmas 4 and 5 specify the input prices that U1 and U2 set when they collude and induce both D1 and D2 to serve customers under VS and under VI. The lemmas also characterize D2's equilibrium profit. Lemma 5 refers to $M \equiv 2\left[\,5 - 2\,f_{11}\left(\,1 - f_{11}\,\right)\right] > 0$.

Lemma 4. *Under VS and collusion, U1 and U2 set input price* $w_{12}^{Sc} = \frac{1}{4}\left[\,2\,a + 2\,c^u - c_1^d - c_2^d\,\right]$ $> c^u$,[29] *and D2 secures equilibrium profit* $\pi_2^{Sc} = \frac{1}{144\,b}\left[\,2\,a - 2\,c^u + 5\,c_1^d - 7\,c_2^d\,\right]^2$.

Lemma 5. *Suppose* $c_1^d > c_2^d$. *Then under VI and collusion, U1 and U2 set input price* $w_{12}^{Ic} = \frac{1}{M}\left\{\,4\,a + 6\,c^u + c_1^d - 5\,c_2^d + f_{11}\left[\,a - 5\,c^u - 2\,c_1^d + c_2^d\,\right] + 4\,c^u\left(f_{11}\right)^2\,\right\} > c^u$, *and D2 secures equilibrium profit* $\pi_2^{Ic} = \frac{1}{9\,b}\left[\,a - w_{12}^{Ic} - 2\,c_2^d + c_1^d - f_{11}\left(\,w_{12}^{Ic} - c^u\,\right)\right]^2$.

The input prices specified in Lemmas 4 and 5 are readily compared.

Conclusion 3. $w_{12}^{Ic} > w_{12}^{Sc}$ *if* $c_1^d > c_2^d$.

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value of f_{12} does not affect D2's choice of q_2 because D2 has no vested interest in the upstream profit of U1 or U2.

[28] It can be shown that U1 and U2 also increase the input price as c_2^d increases under VI and collusion. The higher input price shifts industry output away from D2 as its downstream cost increases.

[29] The subscript "12" here (and below) denotes strictly positive equilibrium output by both D1 and D2. The superscript "Sc" denotes vertical <u>s</u>eparation and upstream <u>c</u>ollusion.

Conclusion 3 reports that when D1 and D2 both serve customers in equilibrium, U1 and U2 increase the collusive input price further above cost under VI than under VS. The higher upstream profit margin endows D1 with a more pronounced competitive advantage over D2. D1's competitive advantage under VI arises because, with $f_{11} > \frac{1}{2}$ (recall Conclusion 2), D1 effectively perceives a lower input price than does D2. The perceived difference in input price increases as the established input price increases. U1 and U2 endow D1 with an increased competitive advantage under VI in order to enhance D1's profit, which U1 values under VI.

It remains to calculate Δ_{L2}^{C}, the difference between D2's loss from upstream collusion under VI and its corresponding loss under VS in the presence of downstream Cournot competition. Formally, $\Delta_{L2}^{C} \equiv \pi_2^{I} - \pi_2^{Ic} - \left(\pi_2^{S} - \pi_2^{Sc} \right) = \pi_2^{Sc} - \pi_2^{Ic}$. The last equality here holds because $\pi_2^{S} = \pi_2^{I}$, from Conclusion 1.

Recall that when U1 and U2 collude, they drive D2 from the market (so D2 secures 0 profit) under VI when $c_2^{d} \geq c_1^{d}$ (Lemma 3). Consequently, because D2 secures the same profit under VI and VS in the absence of collusion, D2's loss from collusion is greater under VI than under VS.

Proposition 1. *If $c_2^{d} \geq c_1^{d}$, then D2's incremental loss from collusion under VI is $\Delta_{L2}^{C} = \pi_2^{Sc} = \frac{1}{144\,b} \left[2\,a - 2\,c^{u} + 5\,c_1^{d} - 7\,c_2^{d} \right]^{2} > 0$.*

The comparison of D2's loss from collusion under VS and under VI is less straightforward when D2 always serves customers in equilibrium (because $c_2^{d} < c_1^{d}$). However, it can be shown that D2's loss is always greater under VI in this case.

Proposition 2. $\Delta_{L2}^{C} > 0$ *if $c_2^{d} < c_1^{d}$.*

Proposition 2 reflects two considerations. First, U1 and U2 increase the collusive input price further above cost (c^{u}) under VI than under VS (Conclusion 3). Second, because the input price exceeds c^{u} (Lemma 5), D1 enjoys a competitive advantage over D2 under VI. The higher input price and the competitive disadvantage that D2 experiences under VI ensure the profit reduction that collusion imposes on D2's is more pronounced under VI than under

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VS.

Propositions 1 and 2 together imply that D2's loss from collusion is always greater under VI than under VS in the presence of downstream Cournot competition.[30]

Corollary. $\Delta^C_{L2} \geq 0$.

4 Hotelling Competition

We now establish that different conclusions can arise in settings where the downstream suppliers engage in price competition and sell differentiated products. We begin by demonstrating that particularly distinct conclusions arise under the standard formulation of Hotelling competition where $N > 0$ consumers are uniformly distributed on the unit interval between D1's location (point 0) and D2's location (point 1). Each consumer derives value v_i from one unit of Di's product ($i \in \{1, 2\}$).[31] A consumer located at point $l \in [0, 1]$ that purchases the product from D1 at price p_1 secures net utility $v_1 - p_1 - tl$, where $t > 0$ reflects a unit transportation cost that all customers incur. If the customer at point l purchases the product from D2 at price p_2, she secures net utility $v_2 - p_2 - t[1 - l]$.

Assumptions 3 – 5 are presumed to hold throughout the ensuing analysis in this section. Assumption 3 helps to ensure that full market coverage prevails in equilibrium (i.e., every consumer purchases one unit of the product), which we presume to be the case. Assumptions 4 and 5 help to ensure that D1 and D2 both serve customers in equilibrium.

Assumption 3. $v_1 + v_2 > 3t + c^d_1 + c^d_2 + 2c^u$.

Assumption 4. $v_i > c^d_i + c^u$ for $i = 1, 2$.

Assumption 5. $\left| v_1 - c^d_1 - \left(v_2 - c^d_2 \right) \right| < 3t$.

To characterize equilibrium retail prices and quantities under both VS and VI, it is

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[30]If $c^d_2 > c^d_1$ and Assumption 2 does not hold, U1 and U2 may drive D2 from the market (so $q^d_2 = 0$ in equilibrium) when they collude both under VS and under VI. Consequently, D2's loss from collusion is the same under VS and under VI (so $\Delta^C_{L2} = 0$).

[31]Consumers value only a single unit of the retail product that D1 and D2 sell. Furthermore, resale is not possible, so each consumer buys at most one unit of the product.

convenient to suppose that D1 chooses p_1 to maximize its downstream profit plus the fraction

$\alpha_1^d \in \{0, 1\}$ of U1's upstream profit (observe that $\alpha_1^d = 0$ under VS and $\alpha_1^d = 1$ under VI).

In this case, D1 chooses p_1 to maximize

$$\left[p_1 - w - c_1^d \right] Q_1^H(p_1, p_2) + \alpha_1^d \left[w - c^u \right] \left[f_{11} Q_1^H(p_1, p_2) + f_{12} Q_2^H(p_2, p_1) \right],$$

taking p_2 as given, where $Q_i^H(p_i, p_j)$ is the equilibrium demand for Di's product when it

sets price p_i and Dj sets price p_j $(i, j \in \{1, 2\}, \ j \neq i)$. D2 chooses p_2 to maximize

$\left[p_2 - w - c_2^d \right] Q_2^H(p_2, p_1)$, taking p_1 as given.

Lemma 6. *Given input price w and supply relations f_{11} and f_{12}, equilibrium prices and*

quantities in the presence of full market coverage are:

$$p_1^* = \frac{1}{3} \left[3t + 3w + 2c_1^d + c_2^d + v_1 - v_2 - 2\alpha_1^d (w - c^u)(f_{11} - f_{12}) \right];$$

$$p_2^* = \frac{1}{3} \left[3t + 3w + 2c_2^d + c_1^d + v_2 - v_1 - \alpha_1^d (w - c^u)(f_{11} - f_{12}) \right];$$

$$Q_1^* \equiv Q_1(p_1^*, p_2^*) = \frac{N}{6t} \left[3t + v_1 - v_2 - c_1^d + c_2^d + \alpha_1^d (w - c^u)(f_{11} - f_{12}) \right]; \ and$$

$$Q_2^* \equiv Q_2(p_2^*, p_1^*) = \frac{N}{6t} \left[3t + v_2 - v_1 - c_2^d + c_1^d - \alpha_1^d (w - c^u)(f_{11} - f_{12}) \right].$$

Corollary. Under VS, $\frac{\partial p_1^*}{\partial w} = \frac{\partial p_2^*}{\partial w} = 1$ and $\frac{\partial Q_1^*}{\partial w} = \frac{\partial Q_2^*}{\partial w} = 0.$

The Corollary to Lemma 6 reflects the fact that under VS with full market coverage,

D1 and D2 both pass along to consumers in the form of a higher equilibrium retail price

the full amount of any increase in a common unit cost (e.g., w) they incur. Consequently,

equilibrium price-cost margins and outputs (and thus profits) do not vary with w.

When U1 and U2 collude under VS, they set w to maximize $\left[w - c^u \right] \left[Q_1^H(\cdot) + Q_2^H(\cdot) \right].$

The prices and quantities specified in Lemma 6 imply that D2's equilibrium profit under VS

(where $\alpha_1^d = 0$) is as specified in Conclusion 4.

Conclusion 4. *D2's equilibrium profit under VS is* $\Pi_2^{S*} = \frac{N}{18t} \left[3t + v_2 - v_1 - c_2^d + c_1^d \right]^2.$

Conclusion 4 implies that D2's equilibrium profit does not vary with w under VS in the

presence of full market coverage. Consequently, upstream collusion that increases the input price does not affect D2's equilibrium profit in this setting.

In contrast, collusion will affect D2's equilibrium profit under VI. To establish this fact, the findings in Lemma 6 can be employed to specify D2's equilibrium profit when D1 and D2 face input price w and supply relations f_{11} and f_{12}.

Lemma 7. *D2's equilibrium profit given w, f_{11}, f_{12}, and α_1^d is[32]* $\Pi_2^*(w) =$

$$\Pi_2^{S*} - \frac{N \alpha_1^d}{18\,t} \left[\,w - c^u\,\right]\left[\,f_{11} - f_{12}\,\right]\left\{\,2\left[\,3\,t + v_2 - v_1 - c_2^d + c_1^d\,\right] - \alpha_1^d\left[\,w - c^u\,\right]\left[\,f_{11} - f_{12}\,\right]\,\right\}.$$

Lemma 7 can be employed to determine D2's equilibrium profit under VI when U1 and U2 do not collude. In this setting, U1 competes vigorously for D2's patronage by undercutting any input price above c^u that U2 offers to D2. This intense competition drives the equilibrium price of the input to cost.

Conclusion 5. *In the absence of collusion under VI, $w_1 = w_2 = c^u$ and D2 secures equilibrium profit* $\Pi_2^*(c^u) = \Pi_2^{S*} = \frac{N}{18t}\left[\,3\,t + v_2 - v_1 - c_2^d + c_1^d\,\right]^2.$

Conclusion 4 implies that D2's profit under VS is Π_2^{S*} both in the presence of collusion and in its absence. Lemma 7 and Conclusion 5 imply that D2's profit under VI is: (i) Π_2^{S*} in the absence of collusion; and (ii) $\Pi_2^*(w^{Ic})$ in the presence of collusion, where w^{Ic} denotes the input price that U1 and U2 set when they collude under VI. Consequently, Δ_{L2}^H, the additional reduction in D2's profit due to collusion that arises because of vertical integration under downstream Hotelling competition is $\Delta_{L2}^H = \Pi_2^{S*} - \Pi_2^*(w^{Ic}) - \left(\Pi_2^{S*} - \Pi_2^{S*}\right) = \Pi_2^{S*} - \Pi_2^*(w^{Ic})$.

Conclusion 6 helps to determine $\Pi_2^*(w^{Ic})$ by characterizing the input price (w) and the supply relations $(f_{11}$ and $f_{12})$ U1 and U2 implement when they collude under VI, and so act to maximize $\left[\,w - c^u\,\right]\left[\,Q_1^H(\cdot) + Q_2^H(\cdot)\,\right] + \left[\,p_1 - w - c_1^d\,\right]Q_1^H(\cdot).$

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[32] The functional dependence of $\Pi_2^*(\cdot)$ on f_{11}, f_{12}, and α_1^d is suppressed, for expositional ease.

Conclusion 6. *In the presence of collusion under VI, $w > c^u$, $f_{12} = 1$, and $f_{11} = 0$.*

Conclusion 6 reports that U1 and U2 increase the input price above c^u when they collude to ensure they secure a strictly positive profit on each unit of the input they sell. They also assign U1 to supply all of D2's input needs and assign U2 to supply all of D1's input needs. These supply relations motivate D1 to compete less aggressively against D2 because a reduction in p_1 reduces D2's output (holding p_2 constant) and thereby reduces U1's profitable sales of the input to D2. D1's incentive to increase D2's output increases as w (and thus U1's upstream profit margin) increases. Consequently, U1 and U2 are able to increase w considerably while ensuring that D2 continues to serve a relatively large number of consumers and therefore continues to purchase a substantial amount of the input.[33]

D1's reduced competitive intensity and the resulting elevated equilibrium prices also enhance D2's equilibrium profit. Consequently, the reduction in D2's profit due to upstream collusion is less pronounced under VI than under VS, as Proposition 3 reports. In fact, as the Corollary to Proposition 3 records for emphasis, upstream collusion actually increases D2's profit under VI in the presence of downstream Hotelling competition.[34]

Proposition 3. $\Delta_{L2}^H < 0$, *so the reduction in D2's profit due to upstream collusion is less pronounced under VI than under VS.*

Corollary. *Upstream collusion increases D2's profit under VI.*

The supply relations identified in Conclusion 6 underlie the increased profit that D2 experiences when U1 and U2 collude under VI. It can be shown that $\Delta_{L2}^H > 0$ if f_{11} is

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[33] Aggregate consumer transportation costs also remain relatively low, so industry surplus is not dissipated unduly, when D2's equilibrium output remains relatively high even though w substantially exceeds c^u.

[34] Gu et al. (2017) show that upstream collusion can benefit downstream suppliers if they hold title to a sufficiently large fraction of upstream profit. Gu et al. (2019) identify conditions under which upstream price collusion increases total industry profit so that, in principle, the upstream suppliers could compensate the (unaffiliated) downstream suppliers that are harmed by the higher input price. The Corollary to Proposition 3 concludes that upstream collusion can benefit an unaffiliated downstream supplier that receives neither a share of the realized upstream profit nor any other direct compensation from the upstream suppliers.

constrained to be 1 and f_{12} is constrained to be 0. In this case, when U1 and U2 increase w^{Ic} above c^u to secure a positive upstream profit margin, D1 competes vigorously against D2 in order to generate (profitable) upstream sales for D1. The increased aggression by D1 reduces D2's profit below the level it achieves under VS and collusion, and thereby ensures $\Delta_{L2}^H > 0$.

5 Linear Demand with Differentiated Products

Hotelling competition with full market coverage is special in part because the aggregate demand for the input is always N, the fixed number of retail customers. We now examine how the findings in Section 4 change when the prevailing input prices and supply relations can affect the total demand for the input in the presence of retail price competition with differentiated products. To do so, we assume that when Di sets price p_i and Dj sets price p_j ($i, j \in \{1, 2\}$, $j \neq i$), the demand for Di's product is:

$$Q_i(p_i, p_j) = a_i - b\, p_i + d\, p_j, \tag{1}$$

where $a_i > 0$ and $b > 0$ are parameters, and where $d \in (0, b)$ is an additional parameter that reflects the extent to which consumers view the two retail products as substitutes.[35]

To focus on settings where D1 and D2 both produce strictly positive output in equilibrium under both VS and VI, Assumptions 6 – 8 are maintained throughout the ensuing analysis. The assumptions ensure that demand is high relative to cost for both products and that the demand and cost structures facing D1 and D2 are not too disparate.

Assumption 6. $A_1 \equiv 2\, b\, a_1 + d\, a_2 + b\, d \left[c_2^d + c^u \right] - \left[2\, b^2 - d^2 \right] \left[c_1^d + c^u \right] > 0$.

Assumption 7. $A_2 \equiv 2\, b\, a_2 + d\, a_1 + b\, d \left[c_1^d + c^u \right] - \left[2\, b^2 - d^2 \right] \left[c_2^d + c^u \right] > 0$.

Assumption 8. $3\, A_1 > A_2$ and $3\, A_2 > A_1$.

In the absence of collusion in this setting, competition drives the two upstream suppliers to price their products at cost. Consequently, equilibrium downstream prices and profits are

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[35]These linear demands can be viewed as being derived from a representative consumer's quadratic utility function (Vives, 1999, chapter 6).

the same under VI and VS. Conclusion 7 records D2's equilibrium profit.[36]

Conclusion 7. *In the absence of collusion,* $w_1 = w_2 = c^u$ *and:*

$$\Pi_2^{S*} = \Pi_2^{I*} = \frac{b}{\left[4 b^2 - d^2\right]^2} \left[2 b a_2 - \left(2 b^2 - d^2\right)\left(c_2^d + c^u\right) + d\left(a_1 + b\left[c_1^d + c^u\right]\right)\right]^2.$$

When U1 and U2 collude under VS, they raise the input price above cost, as Conclusion 8 reports.[37]

Conclusion 8. *Under collusion and VS, U1 and U2 set input price* $w^{Sc} = c^u +$

$\frac{a_1 + a_2 - [b - d]\left[2 c^u + c_1^d + c_2^d\right]}{4[b - d]} > c^u$. *D2's corresponding equilibrium profit is:*

$$\Pi_2^{Sc*} = \frac{b}{16\left[4 b^2 - d^2\right]^2} \left\{\left[6 b - d\right] a_2 - \left[2 b - 3 d\right] a_1 - 2\left[2 b + d\right]\left[b - d\right] c^u \right.$$
$$\left. + \left[2 b^2 + 3 b d - d^2\right] c_1^d - \left[6 b^2 + b d - 3 d^2\right] c_2^d\right\}^2.$$

Conclusion 9 characterizes the input price and the supply relations that U1 and U2 implement when they collude under VI.[38] The Conclusion refers to $\hat{f}_1 \equiv f_{11} - \frac{d}{b} f_{12}$, which is the net rate at which U1's sales of the input increase as D1's downstream output increases under VI.[39] D1 competes more aggressively as $\hat{f}_1$ increases under VI because each increment in D1's downstream output generates more sales of the input for U1.

Conclusion 9. *Under collusion and VI,* $w^{Ic} > c^u$ *and* $\hat{f}_1 > 0$.[40] *In addition,* $\frac{\partial \hat{f}_1}{\partial a_1} < 0$, $\frac{\partial \hat{f}_1}{\partial a_2} > 0$, $\frac{\partial \hat{f}_1}{\partial c_1^d} > 0$, *and* $\frac{\partial \hat{f}_1}{\partial c_2^d} < 0$.

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[36] D2's profit given p_1, p_2 and w is $\left[p_2 - w - c_2^d\right] Q_2(p_2, p_1)$. Unless otherwise noted, the notation in this section parallels the notation employed in Section 4. Specifically, w_i denotes the input price that Ui sets in the absence of collusion $(i = 1, 2)$. In addition, Π_i^{S*} and Π_i^{I*}, respectively, denote the equilbrium profit of Di under VS and under VI in the absence of collusion.

[37] When U1 and U2 collude under VS, they set w to maximize $\left[w - c^u\right]\left[Q_1(\cdot) + Q_2(\cdot)\right]$.

[38] When U1 and U2 collude under VI, they set w, f_{11}, and f_{12} to maximize $\left[w - c^u\right]\left[Q_1(\cdot) + Q_2(\cdot)\right] + \left[p_1 - w - c_1^d\right] Q_1(\cdot)$. Given p_2, w, f_{11}, and f_{12}, D1 chooses p_1 to maximize $\left[p_1 - w - c_1^d\right] Q_1(p_1, p_2) + \left[w - c^u\right]\left[f_{11} Q_1(\cdot) + f_{12} Q_2(\cdot)\right]$.

[39] From (1), $\frac{d}{b} = \frac{\partial Q_2}{\partial p_1} / \left|\frac{\partial Q_1}{\partial p_1}\right| = \left|\frac{\partial Q_2}{\partial Q_1}\right|$. Therefore, $f_{11} - \frac{d}{b} f_{12}$ is the difference between the rate at which U1's delivery of the input to D1 increases as Q_1 increases and the rate at which U1's delivery of the input to D2 declines as Q_1 increases due to the corresponding reduction in Q_2.

[40] The proof of Conclusion 9 demonstrates that f_{11} and f_{12} are not uniquely determined under collusion and VI. Only w^{Ic} and $\hat{f}_1$ are uniquely determined.

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Conclusion 9 reflects the following considerations. U1 and U2 increase w^{Ic} above c^u to ensure they receive strictly positive profit on each unit of the input they sell to D2. U1 and U2 also set $\hat{f}_1 > 0$ to ensure D1 enjoys a competitive advantage over D2 under VI in the sense that D1 effectively perceives it faces a lower input price than D2 faces (w^{Ic}). This competitive advantage serves to increase D1's profit, which U1 values.

To understand the additional findings in Conclusion 9, it is helpful to consider what we term "D2's relative competitive strength," which is the difference between the demand for D2's product and the demand for D1's product when both firms price their products at marginal cost, i.e., $\widehat{Q}_2 - \widehat{Q}_1$, where $\widehat{Q}_i \equiv Q_i(c_i^d + w, c_j^d + w)$ for $i, j \in \{1, 2\}$ $(j \neq i)$. The magnitude of the competitive advantage that U1 and U2 bestow upon D1 under VI by increasing $\hat{f}_1$ depends in part on D2's relative competitive strength. As this strength increases, U1 and U2 can increase D1's competitive advantage without diminishing D2's demand for the input unduly. Consequently, U1 and U2 increase $\hat{f}_1$ as D2's relative competitive strength increases (e.g., as a_2 increases, c_2^d declines, a_1 declines, or c_1^d increases)[41] in order to bolster D1's profit.[42]

The impact of the prevailing industry structure on D2's loss from collusion depends in part on the extent to which U1 and U2 increase the price of the input above its production cost under VS and VI. Under VS, the unit costs of D1 and D2 both increase on a dollar-for-dollar basis as w^{Sc} increases. Under VI, D2's unit cost increases with w^{Ic} on a dollar-for-dollar basis whereas D1's perceived unit cost increases less rapidly with w^{Ic} because $\hat{f}_1 > 0$.[43] Consequently, when they decide how far above c^u to set w^{Ic} under VI, U1 and U2

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[41] Recall from equation (1) that $\widehat{Q}_i = a_i - b\left[c_i^d + w\right] + d\left[c_j^d + w\right]$ for $i, j \in \{1, 2\}$ $(j \neq i)$.

[42] As the products of D1 and D2 become highly differentiated (so $d \to 0$), D1 and D2 effectively do not compete. Consequently, U1 and U2 (nearly) eliminate the distortions that bestow a competitive advantage upon D1. Specifically, U1 supplies (nearly) all of D1's demand for the input (so $\lim_{d \to 0} \hat{f}_1 = \lim_{d \to 0} f_{11} = 1$). This supply arrangement ensures that D1 perceives its unit cost of the input to be (approximately) c^u, thereby (nearly) avoiding the standard double marginalization problem (Spengler, 1950). Furthermore, as $d \to 0$, the collusive input price approaches the price that a profit-maximizing upstream supplier would charge to a monopoly downstream supplier with D2's cost structure (i.e., $\lim_{d \to 0} w^{Ic} = c^u + \frac{a_2 - b\left[c_2^d + c^u\right]}{2b}$). The proofs of these findings appear in the proof of Conclusion 9 in Bet et al. (2019).

[43] Recall that when $\hat{f}_1 = 1$, D1 perceives its unit cost to be $c^u + c_1^d$, regardless of the value of w^{Ic}.

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are particularly concerned with the impact of an increase in w^{Ic} on D2's output, and thus on D2's demand for the input. This concern leads U1 and U2 to set w^{Ic} below w^{Sc} when D2 is a relatively weak competitor (i.e., when D2's relative competitive strength is negative) and to set w^{Ic} above w^{Sc} when D2 is a relatively strong competitor (i.e., when D2's relative competitive strength is positive).

Conclusion 10. *When collusion prevails,* $w^{Ic} \lesseqqgtr w^{Sc} \Leftrightarrow \widehat{Q}_2 \lesseqqgtr \widehat{Q}_1$.

Conclusions 9 and 10 imply that when D2 is a relatively strong competitor, its loss from collusion will be more pronounced under VI than under VS for two reasons. First, D2 faces a higher input price under VI than under VS when it is a relatively strong competitor (Conclusion 10). Second, because U1 and U2 set $w^{Ic} > c^u$ and $\hat{f}_1 > 0$ under VI (Conclusion 9), D2 experiences a competitive disadvantage under VI that it does not face under VS.

This finding is recorded formally in Proposition 4. The proposition refers to $\Delta_{L2}^l = \Pi_2^{I*} - \Pi_2^{Ic*} - \left(\Pi_2^{S*} - \Pi_2^{Sc*} \right)$, which is the difference between the profit reduction that collusion imposes on D2 under VI and under VS in the presence of downstream price competition with linear demand and differentiated products.[44]

Proposition 4. $\Delta_{L2}^l > 0$ *if* $\widehat{Q}_2 \geq \widehat{Q}_1$.

D2's loss from collusion will also be more pronounced under VI than under VS even when D2 is a relatively weak competitor if the products of D1 and D2 are sufficiently homogeneous. In the presence of highly homogeneous products, the competitive advantage that D1 enjoys under VI constrains D2's profit relatively severely, much as in the setting with downstream Cournot competition where D1 and D2 sell identical products.

Proposition 5. $\lim_{d \to b} \Delta_{L2}^l > 0$.

In contrast, D2's loss from collusion will be less pronounced under VI than under VS

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[44]The superscript "*l*" here denotes the present setting with l̲inear demand and differentiated products. Recall that Π_2^{Ic*} and Π_2^{Sc*}, respectively, denote the equilibrium profit of D2 under VI and under VS in the presence of collusion in this setting.

when D2 is a relatively weak competitor and the products of D1 and D2 are sufficiently highly differentiated. When D2 is a relatively weak competitor, U1 and U2 set a lower input price under VI than under VS (Conclusion 10). When the products of D1 and D2 are highly differentiated (so d is sufficiently small), the competitive interaction between D1 and D2 effectively is limited, so the competitive disadvantage that D2 experiences under VI is relatively inconsequential. Therefore, when d is sufficiently small and D2 is a relatively weak competitor, the dominant effect of VI is to reduce the collusive input price that D2 faces. Consequently, VI reduces D2's loss from collusion.

Proposition 6. $\Delta_{L2}^l < 0$ if $\widehat{Q}_2 < \widehat{Q}_1$ and d is sufficiently small.

6 Conclusions

We have examined the impact of vertical integration (VI) on the losses that collusion between two upstream suppliers, U1 and U2, imposes on the downstream supplier (D2) that has no affiliation with either U1 or U2. We found that the impact varies with the nature and the intensity of the prevailing downstream competition. VI tends to increase D2's loss from collusion in the presence of relatively intense downstream competition (e.g., when the downstream products are relatively homogeneous) or when D2 is a relatively strong competitor. In contrast, VI can reduce D2's loss from collusion when D2 is a relatively weak competitor and downstream competition is limited (e.g., when the downstream products are highly differentiated) or when the market demand for the downstream product is highly price inelastic (as under Hotelling competition with full market coverage).

The impact of the vertical integration of U1 and downstream supplier D1 on D2's loss from collusion stems in part from the manner in which U1 and U2 structure supply relations when they collude. To illustrate, under downstream Hotelling competition with full market coverage, U1 is assigned to supply all of the input demanded by D2 and U2 supplies all of the input demanded by D1. These supply relations reduce the intensity of retail price competition under VI by ensuring that if D1 reduces its price to attract customers from

D2, U1's (profitable) sales of the input to D2 decline. This deterrent to aggressive price competition increases the profit of all industry suppliers, including D2.

We have employed tractable functional forms to facilitate sharp analytic predictions about when VI will increase D2's loss from collusion and when it will reduce this loss. However, the basic forces at play in the models we analyzed will arise more generally. Consequently, the key qualitative conclusions drawn from our analysis seem likely to persist when more general demand and cost structures are considered.

As noted in the Introduction, we have taken as given the ability of U1 and U2 to implement a common input price and structure supply relations with downstream producers. Future research might consider the changes that arise when limited commitment powers, bargaining frictions, limited information, or a more restrictive set of policy instruments impede collusion.[45] Future research might also consider different forms of collusion. For instance, U1 and U2 might commit to share realized profit in addition to (or instead of) setting a collusive input price. In this event, D1 would consider the impact of its downstream actions on the upstream profit of both U1 and U2 under VI.

These variations and others may alter the details of our analysis. However, they seem unlikely to change our key qualitative conclusions,[46] which include the following two. First, the impact of VI on the losses that upstream collusion imposes on an unaffiliated downstream supplier varies in predictable, but somewhat subtle, ways with the nature and intensity of downstream competition. Second, the losses from upstream collusion are affected not only by the level at which the input price is set, but also by the supply relations that are established between upstream and downstream producers.

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[45] A more expansive set of policy instruments (e.g., discriminatory collusive input prices and nonlinear structuring of supply relations) might also be considered. Future research might also consider differentiated inputs, alternative sources of substitute inputs, endogenous industry participation, and endogenous vertical relations among industry suppliers.

[46] The optimal collusive supply relations may change as cost structures change. For example, if U1 can supply the input to D1 at much lower cost than U2, then U2 may not be assigned to supply all of D1's demand for the input under VI and downstream Hotelling competition.

Appendix

This Appendix sketches the proofs of all lemmas, conclusions, propositions, and corollaries. Detailed proofs are available in Bet et al. (2019).

Proof of Lemma 1

The proof follows from Findings 1A – 1C. For convenience, the proof: (i) assumes Di purchases the input from Ui ($i = 1, 2$) when indifferent between purchasing the input from U1 and U2; and (ii) focuses on settings where D1 and D2 both produce strictly positive output in equilibrium.

Finding 1A. *Suppose D2 buys the input from U2 at unit price w_2 whereas D1 secures the input from U1. Then in the absence of collusion under VI, the combined equilibrium profit of U1 and D1 is*:

$$\Pi_{12}(w_2) = \frac{1}{9\,b} \left[a - 2 \left(c^u + c_1^d \right) + w_2 + c_2^d \right]^2. \tag{2}$$

Proof. Standard techniques reveal that the equilibrium outputs of D1 and D2 are:

$$q_1^d = \frac{1}{3\,b} \left[a - 2 \left(c^u + c_1^d \right) + w_2 + c_2^d \right] \quad \text{and} \tag{3}$$

$$q_2^d = \frac{1}{3\,b} \left[a - 2 \left(w_2 + c_2^d \right) + c^u + c_1^d \right]. \tag{4}$$

$$\Rightarrow \quad P(q_1^d, q_2^d) = a - \frac{1}{3} \left[2\,a - c^u - c_1^d - w_2 - c_2^d \right] = \frac{1}{3} \left[a + c^u + c_1^d + w_2 + c_2^d \right] \tag{5}$$

$$\Rightarrow \quad P(q_1^d, q_2^d) - c^u - c_1^d = \frac{1}{3} \left[a - 2 \left(c^u + c_1^d \right) + w_2 + c_2^d \right]. \tag{6}$$

(3) and (6) imply that the combined profit of U1 and D1 is as specified in (2). $\square$

Finding 1B. *Suppose D1 and D2 secure the input from U1 at unit price w_1. Then in the absence of collusion under VI, the combined equilibrium profit of U1 and D1 is*:

$$\Pi_{11}(w_1) = \frac{w_1 - c^u}{3\,b} \left[a - 2 \left(w_1 + c_2^d \right) + c^u + c_1^d \right] + \frac{1}{9\,b} \left[a - 2 \left(c^u + c_1^d \right) + w_1 + c_2^d \right]^2. \tag{7}$$

Proof. (3) and (4) imply that the under the specified conditions, the equilibrium outputs of D1 and D2 are:

$$q_1^d = \frac{1}{3\,b} \left[a - 2 \left(c^u + c_1^d \right) + w_1 + c_2^d \right] \quad \text{and} \quad q_2^d = \frac{1}{3\,b} \left[a - 2 \left(w_1 + c_2^d \right) + c^u + c_1^d \right]. \tag{8}$$

Therefore, the equilibrium downstream price is:

$$P(q_1^d, q_2^d) = a - \frac{1}{3} \left[2\,a - c^u - c_1^d - w_1 - c_2^d \right] = \frac{1}{3} \left[a + c^u + c_1^d + w_1 + c_2^d \right]$$

$$\Rightarrow \quad P(q_1^d, q_2^d) - c^u - c_1^d = \frac{1}{3} \left[a - 2 \left(c^u + c_1^d \right) + w_1 + c_2^d \right]. \tag{9}$$

(8) and (9) imply that the combined profit of U1 and D1 is as specified in (7). $\square$

Finding 1C. *In the absence of collusion under VI, D2 purchases the input from U2 at unit price $w_2 = c^u$ and D1 secures the input from U1.*

<u>Proof</u>. Suppose U2 sets $w_2 > c^u$ at a level that generates positive equilibrium output for D2 if D2 purchases the input from U2 and D1 secures the input from U1. Then if U1 sets $w_1 \geq w_2$, D2 will purchase the input from U2 and D1 will secure the input from U1. Consequently, the combined profit of U1 and D1 will be $\Pi_{12}(w_2)$, as specified in (2). If U1 sets w_1 just below w_2, D2 will purchase the input from U1 and D1 will secure the input from U1. Consequently, the combined profit of U1 and D1 will be nearly $\Pi_{11}(w_2)$, as specified in (7). (2) and (7) imply:

$$\Pi_{11}(w_2) > \Pi_{12}(w_2) \quad \Leftrightarrow \quad \frac{w_1 - c^u}{3b} \left[a - 2\left(w_2 + c_2^d\right) + c^u + c_1^d \right] > 0. \tag{10}$$

The last inequality in (10) reflects (4). Consequently, U1 will find it most profitable to slightly undercut any such $w_2 > c^u$. In response, U2 will find it most profitable to match the lower price set by U1. Therefore, the equilibrium value of w_2 cannot exceed c^u.

U2 will secure negative profit if it sets $w_2 < c^u$ and either D1 or D2 purchases the input from U2. In contrast, U2 can secure zero profit in equilibrium by setting $w_2 = c^u$. This is the case because U1 will not reduce w_1 below c^u in response. Doing so would both reduce D1's downstream profit and cause U1 to incur a loss on its sale of the input to D2. Therefore, the equilibrium outcome is as specified. $\square$

The specified expressions for q_1^d and q_2^d follow from (3) and (4) (because $w_2 = c^u$) and from straightforward proofs by contradiction that $q_1^d > 0$ and $q_2^d > 0$ in equilibrium. The proof of the corresponding findings under VS is similar (but more straightforward). $\blacksquare$

Proof of Conclusion 1

Standard techniques reveal that the equilibrium outputs of Dj and Di are:

$$q_j^d = \frac{1}{3b} \left[a - w - 2c_j^d + c_i^d + \left(2\alpha_j^d f_{jj}^u - \alpha_i^d f_{ii} \right)(w - c^u) \right] \quad \text{and} \tag{11}$$

$$q_i^d = \frac{1}{3b} \left[a - w - 2c_i^d + c_j^d + \left(2\alpha_i^d f_{ii} - \alpha_j^d f_{jj}^u \right)(w - c^u) \right] \tag{12}$$

$$\Rightarrow \quad q_i^d + q_j^d = \frac{1}{3b} \left[2a - 2w - c_i^d - c_j^d + \left(\alpha_i^d f_{ii} + \alpha_j^d f_{jj}^u \right)(w - c^u) \right] \tag{13}$$

$$\Rightarrow \quad P(q_i^d, q_j^d) = \frac{1}{3} \left[a + 2w + c_i^d + c_j^d - \left(\alpha_i^d f_{ii} + \alpha_j^d f_{jj}^u \right)(w - c^u) \right] \tag{14}$$

$$\Rightarrow \quad P(q_i^d, q_j^d) - w - c_i^d = \frac{1}{3} \left[a - w - 2c_i^d + c_j^d - \left(\alpha_i^d f_{ii} + \alpha_j^d f_{jj}^u \right)(w - c^u) \right]. \tag{15}$$

(12) and (15) imply:

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$$\left[P(q_i^d, q_j^d) - w - c_i^d\right] q_i^d \;=\; \frac{1}{9\,b}\left[a - w - 2\,c_i^d + c_j^d + \left(2\,\alpha_i^d\,f_{ii} - \alpha_j^d\,f_{jj}^u\right)(w - c^u)\right]$$

$$\cdot \left[a - w - 2\,c_i^d + c_j^d - \left(\alpha_i^d\,f_{ii} + \alpha_j^d\,f_{jj}^u\right)(w - c^u)\right]. \qquad (16)$$

The Conclusion follows from (16) because: (i) $\alpha_1^d = \alpha_2^d = 0$ under VS; and (ii) $\alpha_1^d = 1$ and $\alpha_2^d = 0$ under VI; and (iii) $w = c^u$ from Lemma 1. $\blacksquare$

Proof of Lemma 2

The proof follows from Findings 2A – 2L.

Finding 2A. *Suppose $q_1^d > 0$ and $q_2^d > 0$ in equilibrium. Then for $Z_2 \neq 0$, U1 and U2 will set collusive input price:*

$$w^* \;=\; \frac{Z_1}{2\,Z_2} \qquad (17)$$

where

$$Z_1 \;\equiv\; 3\left[2\,a - c_1^d - c_2^d\right] + 6\,c^u\left[1 - \left(\alpha_1^d\,f_{11} + \alpha_2^d\,f_{22}\right)\right]$$

$$- \alpha_1^u\left[a - 2\,c_1^d + c_2^d\right]\left[2 + 2\,\alpha_2^d\,f_{22} - \alpha_1^d\,f_{11}\right]$$

$$+ \alpha_1^u\,c^u\left[2\,\alpha_1^d\,f_{11} - \alpha_2^d\,f_{22} + \left(\alpha_1^d\,f_{11} + \alpha_2^d\,f_{22}\right)\left(4\,\alpha_1^d\,f_{11} - 2\,\alpha_2^d\,f_{22} - 1\right)\right]$$

$$- \alpha_2^u\left[a - 2\,c_2^d + c_1^d\right]\left[2 + 2\,\alpha_1^d\,f_{11} - \alpha_2^d\,f_{22}\right]$$

$$+ \alpha_2^u\,c^u\left[2\,\alpha_2^d\,f_{22} - \alpha_1^d\,f_{11} + \left(\alpha_1^d\,f_{11} + \alpha_2^d\,f_{22}\right)\left(4\,\alpha_2^d\,f_{22} - 2\,\alpha_1^d\,f_{11} - 1\right)\right], \qquad (18)$$

and

$$Z_2 \;\equiv\; 3\left[2 - \left(\alpha_1^d\,f_{11} + \alpha_2^d\,f_{22}\right)\right]$$

$$- \left[1 + \alpha_1^d\,f_{11} + \alpha_2^d\,f_{22}\right]\left[\alpha_1^u\left(1 + \alpha_2^d\,f_{22} - 2\,\alpha_1^d\,f_{11}\right) + \alpha_2^u\left(1 + \alpha_1^d\,f_{11} - 2\,\alpha_2^d\,f_{22}\right)\right]. \qquad (19)$$

<u>Proof.</u> (13) and (16) imply that if $q_1^d > 0$ and $q_2^d > 0$ in equilibrium, then in the presence of collusion, U1 and U2 act to maximize:

$$J \;=\; \frac{1}{9\,b}\left\{\alpha_1^u\left[a - w - 2\,c_1^d + c_2^d + \left(2\,\alpha_1^d\,f_{11} - \alpha_2^d\,f_{22}\right)(w - c^u)\right]\right.$$

$$\cdot \left[a - w - 2\,c_1^d + c_2^d - \left(\alpha_1^d\,f_{11} + \alpha_2^d\,f_{22}\right)(w - c^u)\right]$$

$$+ \alpha_2^u\left[a - w - 2\,c_2^d + c_1^d + \left(2\,\alpha_2^d\,f_{22} - \alpha_1^d\,f_{11}\right)(w - c^u)\right]$$

$$\cdot \left[a - w - 2\,c_2^d + c_1^d - \left(\alpha_2^d\,f_{22} + \alpha_1^d\,f_{11}\right)(w - c^u)\right]$$

$$\left. + 3\left[w - c^u\right]\left[2\,a - 2\,w - c_1^d - c_2^d + \left(\alpha_1^d\,f_{11} + \alpha_2^d\,f_{22}\right)(w - c^u)\right]\right\}. \qquad (20)$$

The Finding follows immediately from identifying the value of w at which $\frac{\partial J}{\partial w} = 0$. $\square$

Finding 2B. *Suppose $q_1^d > 0$ and $q_2^d > 0$ in equilibrium. Then under VS and collusion, U1 and U2 set input price*:

$$w_{12}^{Sc} = \frac{1}{4}\left[\, 2\,a + 2\,c^u - c_1^d - c_2^d \,\right] \;>\; c^u \tag{21}$$

and secure payoff:

$$J_{12}^{Sc} = \frac{1}{24\,b}\left[\, 2\,a - 2\,c^u - c_1^d - c_2^d \,\right]^2. \tag{22}$$

Proof. (17), (18), and (19) imply that under VS:

$$Z_1 = 3\left[\, 2\,a + 2\,c^u - c_1^d - c_2^d \,\right] \quad \text{and} \quad Z_2 = 6$$

$$\Rightarrow \quad w_{12}^{Sc} - c^u = \frac{1}{4}\left[\, 2\,a - 2\,c^u - c_1^d - c_2^d \,\right] \;>\; 0. \tag{23}$$

(12) and (21) imply that the equilibrium outputs of D1 and D2 are:

$$q_1^d = \frac{1}{3\,b}\left[\, a - w_{12}^{Sc} - 2\,c_1^d + c_2^d \,\right] = \frac{1}{12\,b}\left[\, 2\,a - 2\,c^u + 5\,c_2^d - 7\,c_1^d \,\right] \tag{24}$$

$$q_2^d = \frac{1}{3\,b}\left[\, a - w_{12}^{Sc} - 2\,c_2^d + c_1^d \,\right] = \frac{1}{12\,b}\left[\, 2\,a - 2\,c^u + 5\,c_1^d - 7\,c_2^d \,\right] \tag{25}$$

$$\Rightarrow \quad q_1^d + q_2^d = \frac{1}{12\,b}\left[\, 4\,a - 4\,c^u - 2\,c_1^d - 2\,c_2^d \,\right] = \frac{1}{6\,b}\left[\, 2\,a - 2\,c^u - c_1^d - c_2^d \,\right]. \tag{26}$$

(23) and (26) imply that U1 and U2's equilibrium payoff is $\Pi^u = \left[\, 2\,a - 2\,c^u - c_1^d - c_2^d \,\right]^2$. $\square$

Finding 2C. *Suppose $q_2^d > 0$ and $q_1^d = 0$ in equilibrium. Then under collusion, U1 and U2 will set input price $w_2^c = \frac{1}{2}\left[a + c^u - c_2^d \right] > c^u$ and secure aggregate upstream profit $J_2^c \equiv \frac{1}{8\,b}\left[a - c^u - c_2^d \right]^2 > 0$.*[47]

Proof. It is readily verified that when $q_1^d = 0$, U1 and U2 will set $w = \frac{1}{2}\left[a + c^u - c_2^d \right]$ to secure maximum profit $\frac{1}{8\,b}\left[a - c^u - c_2^d \right]^2$. $\square$

Finding 2D. *Suppose $q_1^d > 0$ and $q_2^d = 0$ in equilibrium. Then under collusion and VS, U1 and U2 will set input price $w_1^{Sc} = \frac{1}{2}\left[a + c^u - c_1^d \right] > c^u$ and secure aggregate upstream profit $J_1^{Sc} \equiv \frac{1}{8\,b}\left[a - c^u - c_1^d \right]^2 > 0$.*

The proof of Finding 2D employs standard maximization procedures, and so is omitted.

Definition. $\hat{w}_i^{Sc} \equiv a - 2\,c_j^d + c_i^d$.

Finding 2E. *Suppose $c_i^d < c_j^d$. Then under VS and collusion, if U1 and U2 choose to ensure $q_j^d = 0$, they will either set: (i) $w = w_i^{Sc}$ and secure payoff J_i^{Sc}; or (ii) $w = \hat{w}_i^{Sc}$ and*

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[47] This conclusion holds under VS and VI.

secure payoff $\hat{J}_i^{Sc} \equiv \frac{1}{b} \left[a - c^u - 2\, c_j^d + c_i^d \right] \left[c_j^d - c_i^d \right]$.

<u>Proof</u>. The conclusion follows from (11) and Findings 2C and 2D, once it is established that U1 and U2 will never set $w > \max\left\{ w_i^{Sc}, \hat{w}_i^{Sc} \right\}$ when they act to ensure $q_j^d = 0$. This fact is readily established by showing that $\frac{\partial \hat{J}_i^{Sc}}{\partial w} \big|_{w = \hat{w}_i^{Sc}} < 0$ under the maintained assumptions. □

Finding 2F. *Suppose* $c_i^d < c_j^d$. *Then* $J_i^{Sc} \geq \hat{J}_i^{Sc}$ *under VS and collusion.*

<u>Proof</u>. U1 and U2 can secure payoff J_i^{Sc} when Dj is exogenously excluded from the market. If Dj can participate in the market, then U1 and U2 may be constrained by having to set an input price that ensures $q_j^d = 0$ in equilibrium. Therefore $J_i^{Sc} \geq \hat{J}_i^{Sc}$. □

Finding 2G. *Suppose that either* $q_1^d = 0$ *or* $q_2^d = 0$ *in equilibrium under vertical separation and collusion. Then* $q_1^d = 0$ *if* $c_1^d > c_2^d$ *whereas* $q_2^d = 0$ *if* $c_2^d > c_1^d$.

<u>Proof</u>. The conclusion follows from Findings 2C and 2D because $J_1^{Sc} \gtreqqless J_2^{c} \Leftrightarrow c_1^d \lesseqqgtr c_2^d$. □

Finding 2H. $q_1^d > 0$ *and* $q_2^d > 0$ *in equilibrium under VS and collusion if* $J_{12}^{Sc} > \max\left\{ J_1^{Sc}, J_2^{c} \right\}$ *and*

$$a - c^u + \min\left\{ \frac{5}{2}\, c_i^d - \frac{7}{2}\, c_j^d,\ \frac{5}{2}\, c_j^d - \frac{7}{2}\, c_i^d \right\} > 0 \ \text{ for } i, j \in \{1, 2\}. \tag{27}$$

<u>Proof</u>. The conclusion follows directly from Findings 2B, 2C, 2D, and 2F because (27) ensures $q_1^d > 0$ and $q_2^d > 0$ in equilibrium when U1 and U2 set $w = w_{12}^{Sc}$. □

Finding 2I.

$$J_{12}^{Sc} > J_2^{c} \ \Leftrightarrow \ \left[a - c^u - c_2^d \right]^2 - 4 \left[a - c^u - c_2^d \right] \left[c_1^d - c_2^d \right] + \left[c_1^d - c_2^d \right]^2 > 0 \tag{28}$$

and

$$J_{12}^{Sc} > J_1^{Sc} \ \Leftrightarrow \ \left[a - c^u - c_1^d \right]^2 - 4 \left[a - c^u - c_1^d \right] \left[c_2^d - c_1^d \right] + \left[c_2^d - c_1^d \right]^2 > 0. \tag{29}$$

<u>Proof</u>. The conclusion follows directly from Findings 2B, 2C, and 2D. □

Finding 2J. $J_{12}^{Sc} \gtreqqless J_1^{Sc} \ \Leftrightarrow \ c_2^d - c_1^d \lesseqqgtr \left[2 - \sqrt{3} \right] \left[a - c^u - c_1^d \right]$.

<u>Proof</u>. Finding 2I implies:
$$J_{12}^{Sc} \gtreqqless J_1^{Sc} \ \Leftrightarrow \ x_1^2 - 4\, x_1\, y_1 + y_1^2 \gtreqqless 0 \tag{30}$$
where $x_1 \equiv c_2^d - c_1^d$ and $y_1 \equiv a - c^u - c_1^d$. The roots of the quadratic equation in (30) are determined by:

$$\frac{1}{2}\left[4\,y_1 \pm \sqrt{16\,y_1^2 - 4\,y_1^2}\,\right] \;=\; 0 \quad\Leftrightarrow\quad 2\,y_1 \pm y_1\,\sqrt{3} \;=\; 0. \tag{31}$$

(31) implies:

$$x_1^2 - 4\,x_1\,y_1 + y_1^2 \;>\; 0 \ \text{ if } \ x_1 \;<\; y_1\,[\,2 - \sqrt{3}\,] \ \text{ or } \ x_1 \;>\; y_1\,[\,2 + \sqrt{3}\,]. \tag{32}$$

It is readily verified that the last inequality in (32) cannot hold because:

$$x_1 \;\geq\; y_1\,[\,2 + \sqrt{3}\,] \quad\Leftrightarrow\quad c_2^d - c_1^d \;\geq\; [\,2 + \sqrt{3}\,]\,[\,a - c^u - c_1^d\,] \;>\; a - c^u - c_1^d. \tag{33}$$

(33) implies $a - c^u - c_2^d < 0$, which violates the maintained assumption. Therefore, (30) and (32) imply:

$$J_{12}^{Sc} \;>\; J_1^{Sc} \ \text{ if } \ x_1 \;<\; y_1\,[\,2 - \sqrt{3}\,] \quad\Leftrightarrow\quad c_2^d - c_1^d \;<\; [\,2 - \sqrt{3}\,]\,[\,a - c^u - c_1^d\,].$$

(31) also implies:

$$x_1^2 - 4\,x_1\,y_1 + y_1^2 \;<\; 0 \ \text{ if } \ x_1 \;\in\; (\,y_1\,[\,2 - \sqrt{3}\,],\, y_1\,[\,2 + \sqrt{3}\,]\,). \tag{34}$$

Observe that:

$$x_1 \;>\; y_1\,[\,2 - \sqrt{3}\,] \quad\Leftrightarrow\quad c_2^d - c_1^d \;>\; [\,2 - \sqrt{3}\,]\,[\,a - c^u - c_1^d\,]. \tag{35}$$

(30), (34), (35), and the contradiction in (33) imply:

$$J_{12}^{Sc} \;<\; J_1^{Sc} \ \text{ if } \ c_2^d - c_1^d \;>\; [\,2 - \sqrt{3}\,]\,[\,a - c^u - c_1^d\,].$$

Finally, (31) implies:

$$x_1^2 - 4\,x_1\,y_1 + y_1^2 \;=\; 0 \ \text{ if } \ x_1 \;=\; y_1\,[\,2 - \sqrt{3}\,] \ \text{ or } \ x_1 \;=\; y_1\,[\,2 + \sqrt{3}\,]. \tag{36}$$

(30), (36), and the contradiction in (33) imply:

$$J_{12}^{Sc} \;=\; J_1^{Sc} \ \text{ if } \ c_2^d - c_1^d \;=\; [\,2 - \sqrt{3}\,]\,[\,a - c^u - c_1^d\,]. \quad \square$$

Finding 2K. $J_{12}^{Sc} \gtreqless J_2^c \Leftrightarrow c_1^d - c_2^d \lesseqgtr [\,2 - \sqrt{3}\,]\,[\,a - c^u - c_2^d\,].$

The proof of Finding 2K parallels the proof of Finding 2J, and so is omitted.

Finding 2L. $q_1^d > 0$ *and* $q_2^d > 0$ *in equilibrium under VS and collusion.*

<u>Proof</u>. First suppose $c_1^d \geq c_2^d$. Then Assumption 2 holds if and only if:

$$c_1^d - c_2^d \;<\; \left[\,2 - \sqrt{3}\,\right]\,[\,a - c^u - c_2^d\,].$$

Finding 2K implies that $J_{12}^{Sc} > J_2^c$ when this inequality holds. Finding 2J implies that $J_{12}^{Sc} > J_1^{Sc}$ when $c_1^d \geq c_2^d$ because $2 - \sqrt{3} > 0$ and because $a - c^u - c_1^d > 0$, by assumption. Therefore, $q_1^d > 0$ and $q_2^d > 0$ in equilibrium under collusion and VS when Assumption 2 holds and $c_1^d \geq c_2^d$.

Now suppose $c_1^d < c_2^d$. Then Assumption 2 holds if and only if:

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$$c_2^d - c_1^d < \left[2 - \sqrt{3}\right] \left[a - c^u - c_1^d\right].$$

Finding 2J implies that $J_{12}^{Sc} > J_1^{Sc}$ when this inequality holds. Finding 2K implies that $J_{12}^{Sc} > J_2^c$ when $c_1^d < c_2^d$. It is also readily verified that (27) holds in this case. Therefore, $q_1^d > 0$ and $q_2^d > 0$ in equilibrium under collusion and VS if Assumption 2 holds and $c_1^d < c_2^d$. $\square$ $\blacksquare$

Proof of Lemma 3

The proof follows from Findings 3A – 3E.

Finding 3A. *Suppose $q_1^d > 0$ and $q_2^d > 0$ in equilibrium. Then under collusion and VI, U1 and U2 will set input price*

$$w_{12}^{Ic} = \frac{1}{M} \left\{ 4a + 6c^u + c_1^d - 5c_2^d + f_{11}\left[a - 5c^u - 2c_1^d + c_2^d\right] + 4c^u\left(f_{11}\right)^2 \right\} > c^u, \quad (37)$$

where $M \equiv 2\left[5 - 2f_{11}\left(1 - f_{11}\right)\right] > 0$

and secure payoff $J_{12}^{Ic} = \frac{1}{18\,b\,M} \left[2\left(\beta_0\right)^2 M + \left(\beta_1 + \beta_0\,f_{11}\right)^2\right]$

where $\beta_0 \equiv a - c^u - 2c_1^d + c_2^d$ *and* $\beta_1 \equiv 4a - 4c^u + c_1^d - 5c_2^d.$ (38)

Proof. (18) and (19) imply that under VI:

$$Z_1 = 4a + 6c^u + c_1^d - 5c_2^d + f_{11}\left[a - 5c^u - 2c_1^d + c_2^d\right] + 4c^u\left(f_{11}\right)^2, \quad \text{and} \quad (39)$$

$$Z_2 = 5 - 2f_{11} + 2\left(f_{11}\right)^2 = 5 - 2f_{11}\left[1 - f_{11}\right]. \quad (40)$$

(17), (39), and (40) imply that U1 and U2 will set input price w_{12}^{Ic}, as specified in (37).

To prove $w_{12}^{Ic} > c^u$, observe that:

$$w_{12}^{Ic} = \frac{1}{M} \left\{ 3\left[2a - c_1^d - c_2^d\right] + 6c^u\left[1 - f_{11}\right] - \left[a - 2c_1^d + c_2^d\right]\left[2 - f_{11}\right]\right.$$
$$\left. + c^u f_1^u\left[1 + 4f_{11}\right]\right\} \quad (41)$$

$$\Rightarrow \quad w_{12}^{Ic} - c^u = \frac{1}{M} \left\{ 4a - 4c^u + c_1^d - 5c_2^d + f_{11}\left[a - c^u - 2c_1^d + c_2^d\right]\right\} > 0. \quad (42)$$

(20) implies that U1 and U2's payoff is:

$$J_{12}^{Ic} = \frac{1}{9\,b\,M^2} \left[\left(\beta_0\right)^2 M^2 + \left(\beta_1 + \beta_0\,f_{11}\right)^2 \frac{M}{2}\right] = \frac{1}{18\,b\,M} \left[2\left(\beta_0\right)^2 M + \left(\beta_1 + \beta_0\,f_{11}\right)^2\right]$$

where β_0 and β_1 are as defined in (38). $\square$

Finding 3B. *Suppose $q_1^d > 0$ and $q_2^d = 0$ in equilibrium. Then under collusion and VI, U1 and U2 secure payoff $J_1^{Ic} \equiv \frac{1}{4b} \left[a - c^u - c_1^d \right]^2 > 0$. They can do so by setting $w = c^u$.*

Proof. When $\alpha_1^d = \alpha_1^u = 1$, U1 and U2's profit-maximizing choice of w is determined by:

$$\left[w - c^u \right] \left[-1 + f_{11} \left(2 - f_{11} \right) \right] = 0 \quad \Rightarrow \quad w = c^u \text{ and/or } f_{11} = 1.$$

It is readily verified that U1 and U2's can secure the maximum feasible profit by setting $w = c^u$. □

Finding 3C. *Suppose $c_2^d \geq c_1^d$. Then under collusion and VI, U1 and U2 set $f_{11} = 1$ and also set an input price that induces D2 to produce no output in equilibrium.*

Proof. Suppose $q_1^d > 0$ and $q_2^d > 0$ in equilibrium. Let λ_{11} denote the Lagrange multiplier associated with the constraint $1 - f_{11} \geq 0$. Then (20) implies that the value of $f_{11} \in [0, 1]$ that maximizes the objective of U1 and U2 when $\alpha_2^u = \alpha_2^d = 0$ (and $q_1^d > 0$ and $q_2^d > 0$ in equilibrium) is determined by:

$$\frac{\partial J}{\partial f_{11}} - \lambda_{11} \leq 0 \quad \text{and} \quad f_{11} \left[\frac{\partial J}{\partial f_{11}} - \lambda_{11} \right] = 0 \tag{43}$$

where:

$$\frac{\partial J}{\partial f_{11}} = \frac{1}{9b} \left\{ \alpha_1^d \alpha_1^u \left[w - c^u \right] \left[a - w - 2 c_1^d + c_2^d - 4 \alpha_1^d f_{11} \left(w - c^u \right) \right] + 3 \alpha_1^d \left[w - c^u \right]^2 \right\}$$

$$\Rightarrow \quad \frac{\partial^2 J}{\partial \left(f_{11} \right)^2} = -\frac{4 \left(\alpha_1^d \right)^2 \alpha_1^u \left[w - c^u \right]^2}{9b} \leq 0. \tag{44}$$

(37) implies that when $\alpha_1^d = \alpha_1^u = 1$:

$$\frac{\partial J}{\partial f_{11}} = \frac{w - c^u}{9b} \left[a + 2 w - 3 c^u - 2 c_1^d + c_2^d - 4 f_{11} \left(w - c^u \right) \right]. \tag{45}$$

(37) implies that when $f_{11} = 1$:

$$w_{12}^{Ic} = \frac{1}{10} \left[5 a + 5 c^u - c_1^d - 4 c_2^d \right] \tag{46}$$

$$\Rightarrow \quad w_{12}^{Ic} - c^u = \frac{1}{2} \left[a - c^u - c_2^d \right] + \frac{1}{10} \left[c_2^d - c_1^d \right] > 0 \text{ if } c_2^d \geq c_1^d. \tag{47}$$

(45) and (46) imply that when $\alpha_1^d = \alpha_1^u = 1$:

$$\left. \frac{\partial J(w_{12}^{Ic})}{\partial f_{11}} \right|_{f_{11} = 1} = \frac{w - c^u}{9b} \left[-2 c_1^d + c_2^d + \frac{1}{5} c_1^d + \frac{4}{5} c_2^d \right] = \frac{w - c^u}{5b} \left[c_2^d - c_1^d \right]. \tag{48}$$

(43), (44), (47), and (48) imply that U1 and U2 will set $f_{11} = 1$ if $c_2^d \geq c_1^d$.

(12) and (46) imply that D2's equilibrium output under the maintained assumptions is:

$$\frac{1}{3b} \left[a - w_{12}^{Ic} - 2 c_2^d + c_1^d - \left(w_{12}^{Ic} - c^u \right) \right] \overset{s}{=} -\frac{6}{5} \left[c_2^d - c_1^d \right] \leq 0. \tag{49}$$

(49) provides a contradiction of the maintained assumption that $q_1^d > 0$ and $q_2^d > 0$ in equilibrium. It is straightforward to verify that when $c_1^d < c_2^d$, U1 and U2 secure a higher equilibrium payoff when $q_1^d > 0$ and $q_2^d = 0$ than when $q_2^d > 0$ and $q_1^d = 0$ or $q_1^d = q_2^d = 0$. Therefore, $q_2^d = 0$ in equilibrium when $c_2^d \geq c_1^d$. $\square$

Finding 3D. $J_{12}^{Ic} > \max\{J_1^{Ic}, J_2^{Ic}\}$ *under VI and collusion if* $c_2^d < c_1^d$.

<u>Proof.</u> We need to show that $J_{12}^{Ic} > \max\{J_1^{Ic}, J_2^c\}$ when $c_2^d < c_1^d$, where, from Findings 2C and 3B:

$$J_1^{Ic} = \frac{1}{4\,b}\left[a - c^u - c_1^d\right]^2 \quad \text{and} \quad J_2^c = \frac{1}{8\,b}\left[a - c^u - c_2^d\right]^2. \tag{50}$$

Define $x_2 \equiv c_1^d - c_2^d$ and $y_1 \equiv a - c^u - c_1^d$. (38) implies:

$$\beta_1 = 4\,\beta_0 + 9\left[c_1^d - c_2^d\right]. \tag{51}$$

(51) and Finding 3A imply that when $q_1^d > 0$ and $q_2^d > 0$ in equilibrium:

$$J_{12}^{Ic} = \frac{1}{2\,b\,M}\left[(\beta_0)^2\left(4 + (f_{11})^2\right) + 2\,\beta_0\left(4 + f_{11}\right)\left(c_1^d - c_2^d\right) + 9\left(c_1^d - c_2^d\right)^2\right]. \tag{52}$$

(38) implies:
$$\beta_0 = a - c^u - c_1^d - \left(c_1^d - c_2^d\right) = y_1 - x_2 > 0. \tag{53}$$

The inequality in (53) holds because $a - c^u - 2\,c_1^d + c_2^d > 0$, by assumption. (52) and (53) imply:

$$J_{12}^{Ic} = \frac{1}{2\,b\,M}\left[y_1^2\left(4 + (f_{11})^2\right) + 2\,y_1\,x_2\,f_{11}\left(1 - f_{11}\right) + \left(5 - 2\,f_{11} + (f_{11})^2\right)x_2^2\right]. \tag{54}$$

First suppose $J_1^{Ic} \geq J_2^c$. Because $\lim_{f_{11} \to 1} M = 10$, (50) and (54) imply:

$$\lim_{f_{11} \to 1}\left(J_{12}^{Ic} - J_1^{Ic}\right) = \frac{y_1^2}{4\,b}\left[\frac{8 + 2}{10} - 1\right] + \frac{1}{20\,b}\left[4\,x_2^2\right] = \frac{x_2^2}{5\,b} > 0. \tag{55}$$

(55) implies there exists a $\widetilde{f}_{11}^u$ sufficiently close to 1 for which $J_{12}^{Ic}(\widetilde{f}_{11}^u) > J_1^{Ic}$. Therefore, because J_1^{Ic} does not vary with f_{11} (from (50)), $J_{12}^{Ic*} \equiv \max_{f_{11} \in (\frac{1}{2}, 1)} J_{12}^{Ic}(f_{11}) \geq J_{12}^{Ic}(\widetilde{f}_{11}^u) > J_1^{Ic}$. Consequently, when $J_1^{Ic} \geq J_2^c$, U1 and U2 will set f_{11} (and w_{12}^{Ic}) to ensure that D1 and D2 both serve customers in equilibrium.

Now suppose $J_2^c \geq J_1^{Ic}$. (52) implies that when $q_1^d > 0$ and $q_2^d > 0$ in equilibrium:

$$J_{12}^{Ic} = \frac{1}{2\,b\,M}\left[(\beta_0)^2\left(4 + (f_{11})^2\right) + 2\,\beta_0\left(4 + f_{11}\right)x_2 + 9\,x_2^2\right]. \tag{56}$$

(38) and (50) imply:

$$J_2^c = \frac{1}{8\,b}\left[a - c^u - 2\,c_1^d + c_2^d + 2(c_1^d - c_2^d)\right]^2 = \frac{1}{8\,b}\left[(\beta_0)^2 + 4\,\beta_0\,x_2 + 4\,x_2^2\right]. \tag{57}$$

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Because $\lim_{f_{11} \to \frac{1}{2}} M = 2\left[5 - \frac{1}{2}\right] = 9$, (56) and (57) imply:

$$\lim_{f_{11} \to 1/2} \left(J_{12}^{Ic} - J_2^c \right) = \frac{1}{2\,b} (\beta_0)^2 \left[\frac{4 + 1/4}{9} - \frac{1}{4} \right] > 0. \tag{58}$$

(58) implies there exists a $\widehat{f}_{11}^u$ sufficiently close to $\frac{1}{2}$ for which $J_{12}^{Ic}(\widehat{f}_{11}^u) > J_2^c$. Therefore, because J_2^c does not vary with f_{11} (from (50)), $J_{12}^{Ic*} \equiv \max_{f_{11} \in (\frac{1}{2},1)} J_{12}^{Ic}(f_{11}) \geq J_{12}^{Ic}(\widehat{f}_{11}^u) > J_2^c$. Consequently, when $J_2^c \geq J_1^{Ic}$, U1 and U2 will set f_{11} (and w_{12}^{Ic}) to ensure that D1 and D2 both serve customers in equilibrium. $\square$

Finding 3E. $q_1^d > 0$ and $q_2^d > 0$ in equilibrium under collusion and VI if $c_2^d < c_1^d$.

<u>Proof</u>. Finding 3D ensures that when $c_2^d < c_1^d$ and VI and collusion prevail, U1 and U2 secure higher profit when $q_1^d > 0$ and $q_2^d > 0$ than when $q_1^d = 0$ or $q_2^d = 0$. It remains to show that $q_2^d > 0$ when $w = w_{12}^{Ic}$.

It can be shown that U1 and U2 will set $f_{11} \in (\frac{1}{2}, 1)$ under VI and collusion when $c_2^d < c_1^d$, $q_1^d > 0$, and $q_2^d > 0$. (See Conclusion 2 below.) From (42):

$$w_{12}^{Ic} - c^u \,|_{f_{11} = \frac{1}{2}} = \frac{1}{2} \left[a - c^u - c_2^d \right]. \tag{59}$$

(11) and (59) imply:

$$q_2^d \,|_{f_{11} = \frac{1}{2}} \stackrel{s}{=} \frac{1}{4} \left[a - c^u \right] - \frac{5}{4} c_2^d + c_1^d \stackrel{s}{=} a - c^u - c_2^d + \left(c_1^d - c_2^d \right) > 0, \text{ and}$$

$$\frac{\partial q_2^d}{\partial f_{11}} = -\left[w^{Ic} - c^u \right] \left[1 - \frac{1 + f_{11}}{M} \left(8 f_{11} - 4 \right) \right] - \frac{1 + f_{11}}{M} \left[a - c^u - 2 c_1^d + c_2^d \right] < 0.$$

It remains to show that $q_2^d > 0$ for f_{11} sufficiently close to 1. It can be shown that $f_{11} \to 1$ as $c_2^d \to c_1^d$. (See Conclusion 2 below.) Therefore, (11) and (42) imply:

$$\lim_{c_2^d \to c_1^d} q_2^d = a - c^u - c_1^d - 2 \left[\frac{1}{2} \right] \left[a - c^u - c_1^d \right] = 0.$$

Because $q_2^d > 0$ when $f_{11} = \frac{1}{2}$, $q_2^d = 0$ when $f_{11} \to 1$, and q_2^d is strictly decreasing in f_{11}, it must be the case that $q_2^d > 0$ for all $f_{11} \in (\frac{1}{2}, 1)$. $\square$ $\blacksquare$

Proof of Conclusion 2

Lemma 3 implies that $q_1^d > 0$ and $q_2^d > 0$ in equilibrium under VI and collusion when $c_1^d > c_2^d$. (20) and (42) imply that in this case, U1 and U2 act to maximize:

$$\left[a - c^u - 2 c_1^d + c_2^d \right]^2 + \frac{M}{2} \left[w_{12}^{Ic} - c^u \right]^2. \tag{60}$$

(37), (43), and (45) imply that the rate at which the objective function of U1 and U2 increases with f_{11} is:

$$\frac{\partial J}{\partial f_{11}} = \frac{1}{9\,b}\left[\,w_{12}^{Ic} - c^u\,\right]\left[\,a + 2\,w_{12}^{Ic} - 3\,c^u - 2\,c_1^d + c_2^d - 4\,f_{11}\left(\,w_{12}^{Ic} - c^u\,\right)\right]$$

$$\overset{s}{=}\ a - c^u - 2\,c_1^d + c_2^d + 2\left[\,1 - 2\,f_{11}\,\right]\left[\,w_{12}^{Ic} - c^u\,\right]\ >\ 0 \ \text{ if } f_{11} \leq \frac{1}{2}\,. \tag{61}$$

(61) implies $f_{11} > \frac{1}{2}$. (42) and (61) imply $\left.\frac{\partial J}{\partial f_{11}}\right|_{f_{11}=1} \overset{s}{=} -18\left[\,c_1^d - c_2^d\,\right] < 0$. If $f_{11} = 1$, then (43) implies $\frac{\partial J}{\partial f_{11}} = \lambda_{11} \geq 0$, which is a contradiction. Therefore, $f_{11} < 1$.

Because f_{11} is in the interior of $[\,0,1\,]$, (60) implies it is determined by:

$$\frac{\partial J_{12}^{Ic}}{\partial f_{11}} = 0 \ \Leftrightarrow\ M\,\frac{\partial\left(\,w_{12}^{Ic} - c^u\,\right)}{\partial f_{11}} + \left[\,w_{12}^{Ic} - c^u\,\right]\left[\,-2 + 4\,f_{11}\,\right] = 0\,. \tag{62}$$

Substituting for $w_{12}^{Ic} - c^u$ from (42) implies $\frac{\partial J_{12}^{Ic}}{\partial f_{11}} = 0 \ \Leftrightarrow\ f_{11} = \frac{a - c^u - c_1^d}{a - c^u - c_2^d}$. It is apparent that $\frac{\partial f_{11}}{\partial c_1^d} < 0$, $\frac{\partial f_{11}}{\partial c_2^d} > 0$, and $\lim_{c_1^d \to c_2^d} f_{11} = 1$. It is also readily verified that $\frac{\partial f_{11}}{\partial c^u} \overset{s}{=} c_2^d - c_1^d < 0$ and $\frac{\partial f_{11}}{\partial a} \overset{s}{=} c_1^d - c_2^d > 0$. ∎

Proof of Lemma 4

Lemma 2 implies that $q_1^d > 0$ and $q_2^d > 0$ in equilibrium under the specified conditions. Therefore, the expression for w_{12}^{Sc} in Lemma 4 reflects (21). Furthermore, (16) and (21) imply that when $q_1^d > 0$ and $q_2^d > 0$ in equilibrium, D2's profit is:

$$\pi_2 = \frac{1}{9\,b}\left[\,a - w_{12}^{Sc} - 2\,c_2^d + c_1^d\,\right]^2 = \frac{1}{144\,b}\left[\,2\,a - 2\,c^u + 5\,c_1^d - 7\,c_2^d\,\right]^2. \quad ∎$$

Proof of Lemma 5

Lemma 3 implies that $q_1^d > 0$ and $q_2^d > 0$ in equilibrium under the specified conditions. Therefore, the expression for w_{12}^{Ic} in Lemma 5 reflects (37) and the expression for π_2^{Ic} follows from (16). ∎

Proof of Conclusion 3

Lemmas 4 and 5 imply that under the specified conditions:

$$w_{12}^{Ic} - w_{12}^{Sc} \overset{s}{=} \left[\,4 + f_{11} - \frac{M}{2}\,\right]\left[\,a - c^u\,\right] - c_1^d\left[\,2\,f_{11} - 1 - \frac{M}{4}\,\right] - c_2^d\left[\,5 - f_{11} - \frac{M}{4}\,\right]$$

$$>\ \left[\,4 + f_{11} - \frac{M}{2}\,\right]\left[\,a - c^u - c_1^d\,\right]\ >\ 0\,.$$

The first inequality here holds because $c_1^d > c_2^d$ by assumption and because $5 - f_{11} - \frac{M}{4} > 0$, since $f_{11} \in (\frac{1}{2}, 1)$ from Conclusion 2. The last inequality here holds because $4 + f_{11} - \frac{M}{2} > 0$ for all $f_{11} \in (\frac{1}{2}, 1)$. ∎

Proof of Proposition 1

Lemma 2 implies that $q_1^d > 0$ and $q_2^d > 0$ under VS and collusion under the specified conditions. Lemma 4 implies that D2's corresponding equilibrium profit is $\pi_2^{Sc} = \frac{1}{144\,b} \left[2\,a - 2\,c^u + 5\,c_1^d - 7\,c_2^d \right]^2$.

Lemma 3 implies that $q_2^d = 0$ under VI and collusion under the specified conditions. D2's corresponding profit is 0. Conclusion 1 implies that in the absence of collusion, D2's equilibrium profit is $\pi_2^S = \pi_2^I$ under VS and VI. Therefore:

$$\Delta_{L2}^C \;=\; \pi_2^I - 0 - \left(\pi_2^S - \pi_2^{Sc} \right) \;=\; \pi_2^I - \pi_2^S + \pi_2^{Sc} \;=\; \pi_2^{Sc}. \quad \blacksquare$$

Proof of Proposition 2

Recall:
$$\Delta_{L2}^C \;\equiv\; \pi_2^I - \pi_2^{Ic} - \left(\pi_2^S - \pi_2^{Sc} \right) \;=\; \pi_2^{Sc} - \pi_2^{Ic}. \tag{63}$$

Because $c_1^d > c_2^d$ by assumption, Lemma 3 implies that $q_1^d > 0$ and $q_2^d > 0$ under VI and collusion. Lemma 2 implies that $q_1^d > 0$ and $q_2^d > 0$ under VS and collusion in this case.

Define $\beta_2 \equiv a - c^u - 2\,c_2^d + c_1^d > 0$. (63) and Lemmas 4 and 5 imply:

$$\Delta_{L2}^C \;=\; \frac{1}{144\,b} \left[2\,a - 2\,c^u + 5\,c_1^d - 7\,c_2^d \right]^2 - \frac{1}{9\,b} \left[a - w_{12}^{Ic} - 2\,c_2^d + c_1^d - f_{11} \left(w_{12}^{Ic} - c^u \right) \right]^2$$

$$\overset{s}{=} \;\frac{1}{4} \left[\beta_2 + \frac{3}{2} \left(c_1^d - c_2^d \right) \right]^2 - \left[\beta_2 - (1 + f_{11}) \left(w_{12}^{Ic} - c^u \right) \right]^2 \;\overset{s}{=}\; \left[B_1 + B_2 \right] \left[B_1 - B_2 \right] \tag{64}$$

where

$$B_1 \;\equiv\; \frac{1}{2} \left[\beta_2 + \frac{3}{2} \left(c_1^d - c_2^d \right) \right] \;>\; 0 \;\; \text{and} \;\; B_2 \;\equiv\; \beta_2 - \left[1 + f_{11} \right] \left[w_{12}^{Ic} - c^u \right] \;>\; 0. \tag{65}$$

$B_2 > 0$ because $q_2^d > 0$ under VI and collusion when $c_1^d > c_2^d$. Therefore, (11) implies:

$$a - w_{12}^{Ic} - 2\,c_2^d + c_1^d - f_{11} \left[w_{12}^{Ic} - c^u \right] \;>\; 0 \;\;\Rightarrow\;\; \beta_2 - \left[1 + f_{11} \right] \left[w_{12}^{Ic} - c^u \right] \;>\; 0.$$

Because $B_1 + B_2 > 0$, (64) implies $\Delta_{L2}^C \overset{s}{=} B_1 - B_2$.

(42) and (65) imply:

$$B_1 - B_2 \;=\; \frac{1}{2} \left[\beta_2 + \frac{3}{2} \left(c_1^d - c_2^d \right) \right] - \beta_2 + \left[1 + f_{11} \right] \left[w_{12}^{Ic} - c^u \right]$$

$$=\; \beta_2 \left[\frac{3 \left(1 + f_{11} \right)}{M} - \frac{1}{2} \right] + \frac{3}{4} \left[c_1^d - c_2^d \right] + \frac{\left[1 + f_{11} \right]^2}{M} \left[a - c^u - 2\,c_1^d + c_2^d \right] \;>\; 0. \quad \blacksquare \tag{66}$$

Proof of Lemma 6

Standard techniques reveal that the demand curve facing Di is, for $i, j \in \{1, 2\}$ ($j \neq i$):

$$Q_i(p_i, p_j) = \frac{N}{2t} \left[t + v_i - p_i - (v_j - p_j) \right]. \tag{67}$$

If Di incurs unit cost c_i and seeks to maximize its downstream profit when Dj ($j \neq i$) sets price p_j, Di will set:

$$p_i = \frac{1}{2} \left[t + c_i + v_i - v_j + p_j \right]. \tag{68}$$

When D1 seeks to maximize its downstream profit plus the fraction α_1^d of U1's upstream profit, D1 will choose p_1 to:

$$\text{Maximize} \quad \left[p_1 - w - c_1^d \right] Q_1(p_1, p_2) + \alpha_1^d \left[w - c^u \right] \left[f_{11} Q_1(p_1, p_2) + f_{12} Q_2(p_2, p_1) \right]$$

$$\Rightarrow \quad \left[p_1 - w - c_1^d \right] \frac{\partial Q_1(\cdot)}{\partial p_1} + Q_1(\cdot) + \alpha_1^d \left[w - c^u \right] \left[f_{11} \frac{\partial Q_1(\cdot)}{\partial p_1} + f_{12} \frac{\partial Q_2(\cdot)}{\partial p_1} \right] = 0. \tag{69}$$

From (67), $\frac{\partial Q_1(\cdot)}{\partial p_1} = -\frac{N}{2t} = -\frac{\partial Q_2(\cdot)}{\partial p_1}$. Therefore, (67), (68), and (69) imply:

$$p_1 = \frac{1}{2} \left[t + w + c_1^d + v_1 - v_2 - \alpha_1^d \left(w - c^u \right) \left(f_{11} - f_{12} \right) + p_2 \right] \tag{70}$$

$$\Rightarrow \quad p_1^* = \frac{1}{3} \left[3t + 3w + 2c_1^d + c_2^d + v_1 - v_2 - 2\alpha_1^d \left(w - c^u \right) \left(f_{11} - f_{12} \right) \right] \quad \text{and} \tag{71}$$

$$p_2^* = \frac{1}{3} \left[3t + 3w + 2c_2^d + c_1^d + v_2 - v_1 - \alpha_1^d \left(w - c^u \right) \left(f_{11} - f_{12} \right) \right] \tag{72}$$

$$\Rightarrow \quad p_1^* - p_2^* = \frac{1}{3} \left[c_1^d - c_2^d + 2v_1 - 2v_2 - \alpha_1^d \left(w - c^u \right) \left(f_{11} - f_{12} \right) \right] \tag{73}$$

$$\Rightarrow \quad p_1^* - w - c_1^d = \frac{1}{3} \left[3t + v_1 - v_2 - c_1^d + c_2^d - 2\alpha_1^d \left(w - c^u \right) \left(f_{11} - f_{12} \right) \right] \quad \text{and}$$

$$p_2^* - w - c_2^d = \frac{1}{3} \left[3t + v_2 - v_1 - c_2^d + c_1^d - \alpha_1^d \left(w - c^u \right) \left(f_{11} - f_{12} \right) \right]. \tag{74}$$

(67) and (73) provide the identified expressions for $Q_1(p_1^*, p_2^*)$ and $Q_2(p_2^*, p_1^*)$. ∎

Proof of the Corollary to Lemma 6

The proof follows from Lemma 6 because $\alpha_1^d = 0$ under VS. ∎

Proof of Conclusion 4

(74) and Lemma 6 imply that D2's equilibrium profit in this setting is:

$$\Pi_2^* = \frac{N}{18t} \left[3t + v_2 - v_1 - c_2^d + c_1^d - \alpha_1^d \left(w - c^u \right) \left(f_{11} - f_{12} \right) \right]^2 \quad ∎ \tag{75}$$

Proof of Lemma 7

D2's profit in this setting is $\frac{N}{18t}\left[\,3\,t + v_2 - v_1 - c_2^d + c_1^d - \alpha_1^d\,(\,w - c^u\,)\,(\,f_{11} - f_{12}\,)\,\right]^2$, which is readily written as specified in the lemma. ∎

Proof of Conclusion 5

The Conclusion follows from Conclusion 4 and Lemma 7, once it is established that $w_1 = w_1 = c^u$ in the absence of collusion. This proof of this finding parallels the proof of Lemma 1. ∎

Proof of Conclusion 6

Let [PCF] denote the problem that U1 and U2 face when they act collusively in the presence of full market coverage under VI. This problem is to choose $f_{11} \geq 0$, $f_{12} \geq 0$, and w to:

$$\text{Maximize } \Pi_c^u = [\,w - c^u\,]\,N + \left[\,p_1^* - w - c_1^d\,\right] Q_1(p_1^*, p_2^*) \tag{76}$$

$$\text{subject to: } 1 - f_{1i} \geq 0 \text{ for } i = 1, 2, \tag{77}$$

where the values of p_1^*, p_2^*, and $Q_1(p_1^*, p_2^*)$ are as specified in Lemma 6, with $\alpha_1^d = 1$.

The remainder of the proof of the Conclusion follows from Findings 6A – 6C.

Finding 6A. $[\,w - c^u\,]\,[\,f_{11} - f_{12}\,] \leq 0$ *at the solution to [PCF].*

Proof. Define $Q_1^* \equiv Q_1(p_1^*, p_2^*)$. Let $\lambda_i \geq 0$ denote the Lagrange multiplier on the constraint in (77) for $i = 1, 2$, and let $\mathcal{L}$ denote the Lagrangean function associated with [PCF]. Then the necessary conditions for a solution to [PCF] include:

$$\mathcal{L}_w = H = 0\,; \tag{78}$$

$$\mathcal{L}_{f_{11}} = -G - \lambda_1 \leq 0\,; \qquad f_{11}\,[\,\mathcal{L}_{f_{11}}\,] = 0\,; \text{ and} \tag{79}$$

$$\mathcal{L}_{f_{12}} = G - \lambda_2 \leq 0\,; \qquad f_{12}\,[\,\mathcal{L}_{f_{12}}\,] = 0\,; \tag{80}$$

where

$$H \equiv N - \alpha_1^d\,\alpha_1^u\,[\,f_{11} - f_{12}\,]\,\frac{N}{18t}\,\left\{\,3\,t + v_1 - c_1^d - \left(v_2 - c_2^d\right) + 4\,\alpha_1^d\,[\,w - c^u\,]\,[\,f_{11} - f_{12}\,]\,\right\}; \tag{81}$$

$$G \equiv \alpha_1^u\,\alpha_1^d\,[\,w - c^u\,]\,\frac{N}{18t}\,\left[\,3\,t + v_1 - c_1^d - \left(v_2 - c_2^d\right) + 4\,\alpha_1^d\,(\,w - c^u\,)\,(\,f_{11} - f_{12}\,)\,\right]. \tag{82}$$

(78) – (82) are readily employed to prove: (i) If $G > 0$, then $f_{11} = 0$, $f_{12} = 1$, and $w - c^u > 0$; (ii) If $G < 0$, then $f_{11} = 1$, $f_{12} = 0$, and $w - c^u < 0$; and (iii) If $G = 0$, then $[\,w - c^u\,]\,[\,f_{11} - f_{12}\,] \leq 0$. □

Finding 6B. $w > c^u$ *at the solution to [PCF] when* $\alpha_1^d > 0$ *and* $\alpha_1^u > 0$.

Proof. The Finding is established by contradiction, using (78) – (82) and Finding 6A. □

Finding 6C. *If* $\alpha_1^d > 0$ *and* $\alpha_1^u > 0$, *then* $w > c^u$, $f_{12} = 1$, *and* $f_{11} = 0$ *at the solution to [PCF].*

Proof. The Finding is established by employing Finding 6A and 6B to establish (by contradiction) that $w > c^u$ and $G > 0$, which implies $f_{11} = 0$ and $f_{12} = 1$. $\square\blacksquare$

Proof of Proposition 3

Lemma 7 implies that $\Delta_{L2}^H = \Pi_2^{S*} - \Pi_2^*(w^{Ic})$

$$= \frac{N\,\alpha_1^d}{18\,t}\left[w^{Ic} - c^u\right]\left[f_{11} - f_{12}\right]\left\{2\left[3\,t + v_2 - v_1 - c_2^d + c_1^d\right] - \alpha_1^d\left[w^{Ic} - c^u\right]\left[f_{11} - f_{12}\right]\right\}$$

$$= -\frac{N\,\alpha_1^d}{18\,t}\left[w^{Ic} - c^u\right]\left\{2\left[3\,t + v_2 - v_1 - c_2^d + c_1^d\right] + \alpha_1^d\left[w^{Ic} - c^u\right]\right\} < 0. \qquad (83)$$

The last equality in (83) holds because $f_{12} = 1$ and $f_{11} = 0$ from Conclusion 6. The inequality in (83) holds because: (i) $w^{Ic} - c^u > 0$ from Conclusion 6; and (ii) $3\,t + v_2 - c_2^d - (v_1 - c_1^d) > 0$ from Assumption 1. $\blacksquare$

Proof of the Corollary to Proposition 3

The increase in D2's profit from collusion under VI is $\Pi_2^*(w^{Ic}) - \Pi_2^{S*} > 0$. The inequality reflects (83). $\blacksquare$

Proof of Conclusion 7

Standard techniques imply that Di's equilibrium price is:

$$p_i^* = \frac{1}{4\,b^2 - d^2}\left\{2\,b\left[a_i + b\left(c_i^d + w\right) - \alpha_i^d\left(w - c^u\right)\left(b\,f_{ii} - d\,f_{ij}\right)\right]\right.$$

$$\left. + d\left[a_j + b\left(c_j^d + w\right) - \alpha_j^d\left(w - c^u\right)\left(b\,f_{jj} - d\,f_{ji}\right)\right]\right\} \qquad (84)$$

$$\Rightarrow p_i^* - w - c_i^d = \frac{1}{4\,b^2 - d^2}\left\{2\,b\,a_i - \left[2\,b^2 - d^2\right]\left[c_i^d + w\right] - \alpha_i^d\,2\,b\left[b\,f_{ii} - d\,f_{ij}\right]\left[w - c^u\right]\right.$$

$$\left. + d\left[a_j + b\left(c_j^d + w\right) - \alpha_j^d\left(w - c^u\right)\left(b\,f_{jj} - d\,f_{ji}\right)\right]\right\}. \qquad (85)$$

(84) implies:

$$Q_i(p_i^*, p_j^*) = \frac{b}{4\,b^2 - d^2}\left\{2\,b\,a_i - \left[2\,b^2 - d^2\right]\left[c_i^d + w\right] + d\left[a_j + b\left(c_j^d + w\right)\right]\right.$$

$$\left. + \left[w - c^u\right]\left[\alpha_i^d\left(2\,b^2 - d^2\right)\left(f_{ii} - \frac{d}{b}\,f_{ij}\right) - \alpha_j^d\,d\left(b\,f_{jj} - d\,f_{ji}\right)\right]\right\}. \qquad (86)$$

(85) and (86) imply:

$$\left[p_i^* - w - c_i^d\right]Q_i(p_i^*, p_j^*)$$

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$$= \frac{b}{[\,4\,b^2 - d^2\,]^2} \{\, 2\,b\,a_i - [\,2\,b^2 - d^2\,]\,[\,c_i^d + w\,] - \alpha_i^d\,2\,b\,[\,b\,f_{ii} - d\,f_{ij}\,]\,[\,w - c^u\,]$$

$$+ \, d\,[\,a_j + b\,(\,c_j^d + w\,) - \alpha_j^d\,(\,w - c^u\,)\,(\,b\,f_{jj} - d\,f_{ji}\,)\,]\,\}$$

$$\cdot \{\, 2\,b\,a_i - [\,2\,b^2 - d^2\,]\,[\,c_i^d + w\,] + d\,[\,a_j + b\,(\,c_j^d + w\,)\,]$$

$$+ \, [\,w - c^u\,]\left[\,\frac{\alpha_i^d}{b}\,(\,2\,b^2 - d^2\,)\,(\,b\,f_{ii} - d\,f_{ij}\,) - \alpha_j^d\,d\,(\,b\,f_{jj} - d\,f_{ji}\,)\,\right]\,\}\,. \tag{87}$$

The expression for Π_2^{*S} in the Conclusion follows immediately from (87). The proof that $w_1 = w_2 = c^u$ parallels the proof of Lemma 1. ∎

Proof of Conclusion 8

(86) implies:
$$Q_1(p_1^*, p_2^*) + Q_2(p_2^*, p_1^*) \;=\; \frac{b}{4\,b^2 - d^2}\,H_{Q1}\,, \quad \text{where} \tag{88}$$

$$H_{Q1} \;=\; [\,2\,b + d\,]\,[\,a_1 + a_2\,] - [\,2\,b + d\,]\,[\,b - d\,]\,[\,c_1^d + c_2^d + 2\,w\,]$$

$$+ \, [\,w - c^u\,]\,[\,2\,b + d\,]\,[\,b - d\,]\,[\,\alpha_1^d\,f_{11} + \alpha_2^d\,f_{22}\,]$$

$$- \, [\,w - c^u\,]\,\frac{d}{b}\,[\,2\,b + d\,]\,[\,b - d\,]\,[\,\alpha_1^d\,f_{12} + \alpha_2^d\,f_{21}\,]\,. \tag{89}$$

(88) and (89) imply $Q_1(p_1^*, p_2^*) + Q_2(p_2^*, p_1^*) = \frac{b}{2\,b - d}\,H_{Q2}\,$, where:

$$H_{Q2} \;\equiv\; a_1 + a_2 - [\,b - d\,]\,[\,c_1^d + c_2^d + 2\,w\,]$$

$$+ \, [\,w - c^u\,]\,[\,b - d\,]\left[\,\alpha_1^d\,f_{11} + \alpha_2^d\,f_{22} - \frac{d}{b}\,(\,\alpha_1^d\,f_{12} + \alpha_2^d\,f_{21}\,)\,\right]\,. \tag{90}$$

(90) implies that under VS and collusion, U1 and U2 act to maximize:

$$\Pi_u^{Sc} \;=\; \frac{b\,[\,w - c^u\,]}{2\,b - d}\,\{\, a_1 + a_2 - [\,b - d\,]\,[\,c_1^d + c_2^d + 2\,w\,]\,\}$$

$$\Rightarrow \quad \frac{\partial \Pi_n^u}{\partial w} \;=\; 0 \;\Leftrightarrow\; 2\,w \;=\; c^u + \frac{a_1 + a_2 - [\,b - d\,]\,[\,c_1^d + c_2^d\,]}{2\,[\,b - d\,]}\,. \tag{91}$$

(91) implies that w^{Sc} is as specified in the Conclusion.

This expression for w^{Sc} and Conclusion 7 imply that D2's equilibrium profit under vertical separation and collusion is:
$$\Pi_2^{Sc*} \;=\; \frac{b}{[\,4\,b^2 - d^2\,]^2}\,(H_3)^2\,, \tag{92}$$

where:
$$H_3 \;=\; \frac{1}{4}\,[\,6\,b - d\,]\,a_2 - \frac{1}{4}\,[\,2\,b - 3\,d\,]\,a_1 - \frac{1}{2}\,[\,2\,b + d\,]\,[\,b - d\,]\,c^u$$

$$+ \, \frac{1}{4}\,[\,2\,b^2 + 3\,b\,d - d^2\,]\,c_1^d - \frac{1}{4}\,[\,6\,b^2 + b\,d - 3\,d^2\,]\,c_2^d\,. \tag{93}$$

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The expression for Π_2^{Sc*} in the Conclusion follows from (92) and (93). ∎

Proof of Conclusion 9

The proof consists of Lemmas 8 – 14.

Lemma 8. *Under VI, the equilibrium downstream profits of D1 and D2 are, respectively:*

$$
\Pi_1^{Id} = \frac{b}{[4b^2 - d^2]^2} \{ 2ba_1 - [2b^2 - d^2][c_1^d + w] + d[a_2 + b(c_2^d + w)]
$$

$$
+ [w - c^u]\frac{1}{b}[2b^2 - d^2][bf_{11} - df_{12}] \}
$$

$$
\cdot \{ 2ba_1 - [2b^2 - d^2][c_1^d + w] + d[a_2 + b(c_2^d + w)]
$$

$$
- 2b[w - c^u][bf_{11} - df_{12}] \}; \quad and \qquad (94)
$$

$$
\Pi_2^{Id} = \frac{b}{[4b^2 - d^2]^2} \{ 2ba_2 - [2b^2 - d^2][c_2^d + w] + d[a_1 + b(c_1^d + w)]
$$

$$
- d[w - c^u][bf_{11} - df_{12}] \}^2. \qquad (95)
$$

<u>Proof</u>. The conclusions follow directly from (87). □

Lemma 9. *Under collusion and VI, U1 and U2 set input price $w^{Ic} = \frac{M_2}{M_1}$, where:*

$$
M_1 \equiv -4[b-d][2b-d][2b+d]^2 + \frac{2}{b}[b-d][bf_{11} - df_{12}][(2b-d)(2b+d)^2]
$$

$$
+ 2[2b^2 - bd - d^2 + 2b(bf_{11} - df_{12})]
$$

$$
\cdot \left[2b^2 - bd - d^2 - \frac{1}{b}(2b^2 - d^2)(bf_{11} - df_{12}) \right], \quad and \qquad (96)
$$

$$
M_2 \equiv -c^u \frac{1}{b}[b-d][2b-d][2b+d]^2[2b - (bf_{11} - df_{12})]
$$

$$
- [2b-d][2b+d]^2 \{ a_1 + a_2 - [b-d][c_1^d + c_2^d + \frac{c^u}{b}(bf_{11} - df_{12})] \}
$$

$$
+ \{ 2ba_1 + d[a_2 + bc_2^d] - [2b^2 - d^2][c_1^d + \frac{c^u}{b}(bf_{11} - df_{12})] \}
$$

$$
\cdot \{ [2b+d][b-d] + 2b[bf_{11} - df_{12}] \}
$$

$$
+ \{ 2ba_1 + d[a_2 + bc_2^d] - [2b^2 - d^2] c_1^d + 2bc^u(bf_{11} - df_{12}) \}
$$

$$
\cdot \{ [2b+d][b-d] - \frac{1}{b}[2b^2 - d^2][bf_{11} - df_{12}] \}. \qquad (97)
$$

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Proof. (94) and (90) imply that under VI and collusion, U1 and U2 act to maximize:

$$[w - c^u][2b - d][2b + d]^2 \{a_1 + a_2 - [b - d][c_1^d + c_2^d + 2w] + [w - c^u][b - d][f_{11} - \frac{d}{b}f_{12}]\}$$

$$+ \{2ba_1 - [2b^2 - d^2][c_1^d + w] + d[a_2 + b(c_2^d + w)] + [w - c^u]\frac{1}{b}[2b^2 - d^2][bf_{11} - df_{12}]\}$$

$$\cdot \{2ba_1 - [2b^2 - d^2][c_1^d + w] + d[a_2 + b(c_2^d + w)]$$

$$- 2b[w - c^u][bf_{11} - df_{12}]\}. \tag{98}$$

Setting the derivative of (98) with respect to w equal to 0 provides:

$$w\{[2b - d][2b + d]^2 [2(b - d)(f_{11} - \frac{d}{b}f_{12}) - 4(b - d)]$$

$$- 2[b - d][2b - d][2b + d]^2 + [b - d][f_{11} - \frac{d}{b}f_{12}]$$

$$+ [bd - 2b(bf_{11} - df_{12}) - (2b^2 - d^2)]$$

$$\cdot [bd + \frac{1}{b}(2b^2 - d^2)(bf_{11} - df_{12}) - (2b^2 - d^2)]$$

$$+ [bd + \frac{1}{b}(2b^2 - d^2)(bf_{11} - df_{12}) - (2b^2 - d^2)]$$

$$\cdot [bd - 2b(bf_{11} - df_{12}) - (2b^2 - d^2)]\}$$

$$- c^u[2b - d][2b + d]^2 \{[b - d][f_{11} - \frac{d}{b}f_{12}] - 2[b - d]\}$$

$$+ [2b - d][2b + d]^2 \{a_1 + a_2 - [b - d][c_1^d + c_2^d] - c^u[b - d][f_{11} - \frac{d}{b}f_{12}]\}$$

$$+ \{2ba_1 - [2b^2 - d^2]c_1^d + d[a_2 + bc_2^d] - c^u\frac{1}{b}[2b^2 - d^2][bf_{11} - df_{12}]\}$$

$$\cdot \{bd - 2b[bf_{11} - df_{12}] - [2b^2 - d^2]\}$$

$$+ \{2ba_1 - [2b^2 - d^2]c_1^d + d[a_2 + bc_2^d] + 2bc^u[bf_{11} - df_{12}]\}$$

$$\cdot \{bd + \frac{1}{b}[2b^2 - d^2][bf_{11} - df_{12}] - [2b^2 - d^2]\} = 0. \tag{99}$$

The coefficient on w in (99) can be shown to be M_1. Tedious calculations reveal that the remaining terms in (99) can be expressed as $-M_2$. □

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Lemma 10. *Under collusion and VI, U1 and U2 set $w^{Ic} > c^u$.*

<u>Proof.</u> From Lemma 9, $w^{Ic} - c^u = \frac{M_2 - c^u M_1}{M_1}$. From (96):

$$M_1 = -2\,[\,b - d\,]\left[1 - \left(f_{11} - \frac{d}{b}\,f_{12}\right)\right][\,2\,b - d\,][\,2\,b + d\,]^2 - 4\,b^2\,[\,2\,b^2 - d^2\,]\left[f_{11} - \frac{d}{b}\,f_{12}\right]^2$$

$$- 2\,[\,b - d\,][\,2\,b + d\,]\left[2\,b^2 + b\,d - d^2\left(f_{11} - \frac{d}{b}\,f_{12}\right)\right] < 0. \tag{100}$$

The inequality in (100) holds because $b > d$, $1 - (f_{11} - \frac{d}{b}\,f_{12}) \geq 1 - f_{11} \geq 0$, and

$$2\,b^2 + b\,d - d^2\left[f_{11} - \frac{d}{b}\,f_{12}\right] \geq 2\,b^2 + b\,d - d^2 > d\,[\,b - d\,] > 0. \tag{101}$$

(96) and (97) can be employed to show:

$$M_2 - c^u M_1 = - [\,2\,b - d\,][\,2\,b + d\,]\,[\,2\,b + d\,]\,\{\,a_1 + a_2 - [\,b - d\,][\,c_1^d + c_2^d + 2\,c^u\,]\,\}$$

$$+ \{\,2\,b\,a_1 + d\,a_2 + b\,d\,[\,c_2^d + c^u\,] - [\,2\,b^2 - d^2\,]\,[\,c_1^d + c^u\,]\,\}$$

$$\cdot\,\{\,2\,[\,2\,b + d\,][\,b - d\,] + d^2\,[\,f_{11} - \frac{d}{b}\,f_{12}\,]\,\}. \tag{102}$$

$A_1 > 0$ and $A_2 > 0$ by assumption. Therefore:

$$A_1 + A_2 = [\,2\,b + d\,][\,a_1 + a_2\,] - [\,2\,b^2 - b\,d - d^2\,]\,[\,c_1^d + c_2^d + 2\,c^u\,]$$

$$= [\,2\,b + d\,]\left[\,a_1 + a_2 - (\,b - d\,)\,(\,c_1^d + c_2^d + 2\,c^u\,)\right] > 0. \tag{103}$$

In addition:

$$A_1 = 2\,b\,a_1 + d\,a_2 + b\,d\,[\,c_2^d + c^u\,] - [\,2\,b^2 - d^2\,]\,[\,c_1^d + c^u\,] > 0. \tag{104}$$

(102), (103), and (104) imply:

$$M_2 - c^u M_1 = - [\,2\,b - d\,][\,2\,b + d\,]\,A_2 + A_1\,\{\,-2\,b\,d - d^2 + d^2\,[\,f_{11} - \frac{d}{b}\,f_{12}\,]\,\}. \tag{105}$$

Because $f_{11} - \frac{d}{b}\,f_{12} \leq 1$:

$$-2\,b\,d - d^2 + d^2\left[f_{11} - \frac{d}{b}\,f_{12}\right] < 0 \tag{106}$$

Because $A_1 > 0$ and $A_2 > 0$ by assumption, (102) – (106) imply that $M_2 - c^u M_1 < 0$. Therefore, because $M_1 < 0$ from (100), $w^{Ic} - c^u = \frac{M_2 - c^u M_1}{M_1} > 0$. □

Under collusion and VI, U1 and U2 choose f_{11} and f_{12} to maximize:

$$\frac{b\,[\,w^{Ic} - c^u\,]}{2\,b - d}\,\{\,a_1 + a_2 - [\,b - d\,]\,[\,c_1^d + c_2^d + 2\,w^{Ic}\,]$$

$$+ [\,w^{Ic} - c^u\,]\,[\,b - d\,]\,[\,f_{11} - \frac{d}{b}\,f_{12}\,]\,\}$$

$$+ \frac{b}{[\, 4\, b^2 - d^2 \,]^2} \{\, 2\, b\, a_1 - [\, 2\, b^2 - d^2 \,]\, [\, c_1^d + w \,] + d\, [\, a_2 + b\, (\, c_2^d + w \,) \,]$$

$$+ [\, w - c^u \,]\, \frac{1}{b}\, [\, 2\, b^2 - d^2 \,]\, [\, b\, f_{11} - d\, f_{12} \,] \}$$

$$\cdot \, \{\, 2\, b\, a_1 - [\, 2\, b^2 - d^2 \,]\, [\, c_1^d + w \,] + d\, [\, a_2 + b\, (\, c_2^d + w \,) \,]$$

$$- \, 2\, b\, [\, w - c^u \,]\, [\, b\, f_{11} - d\, f_{12} \,] \}. \qquad (107)$$

where w^{Ic} is as specified in Lemma 9. (This conclusion follows directly from (94), (90), and Lemma 9.) Therefore, when U1 and U2 collude under VI, their problem [P-VIc], is:

$$\underset{w,\, f_{11} \geq 0,\, f_{12} \geq 0}{\text{Maximize}} \quad \Pi^u(w, f_{11}, f_{12})$$

$$\text{subject to:} \quad 1 - f_{11} \, \geq \, 0 \quad \text{and} \quad 1 - f_{12} \, \geq \, 0\,, \qquad (108)$$

where $\Pi^u(w, f_{11}, f_{12}) \; =$

$$\frac{b\, [\, w - c^u \,]}{2\, b - d} \{\, a_1 + a_2 - [\, b - d \,]\, [\, c_1^d + c_2^d + 2\, w \,] + [\, w - c^u \,]\, [\, b - d \,]\, [\, f_{11} - \frac{d}{b}\, f_{12} \,] \}$$

$$+ \frac{b}{[\, 4\, b^2 - d^2 \,]^2} \{\, A_1 + [\, w - c^u \,]\, \left[b\, d + (\, 2\, b^2 - d^2 \,)\, \left(f_{11} - \frac{d}{b}\, f_{12} - 1 \right) \right] \}$$

$$\cdot \, \{\, A_1 - [\, w - c^u \,]\, \left[-d^2 - b\, d + 2\, b^2 \left(f_{11} - \frac{d}{b}\, f_{12} + 1 \right) \right] \}. \qquad (109)$$

(109) follows from (107) because:

$$2\, b\, a_1 - [\, 2\, b^2 - d^2 \,]\, [\, c_1^d + w \,] + d\, [\, a_2 + b\, (\, c_2^d + w \,) \,]$$

$$= \; A_1 - [\, w - c^u \,]\, [\, 2\, b^2 - b\, d - d^2 \,]. \qquad (110)$$

Differentiating (109) provides $\frac{\partial \Pi^u(\cdot)}{\partial f_{11}} \, = \, b\, E$, where:

$$E \, \equiv \, \frac{[\, w - c^u \,]^2\, [\, b - d \,]}{2\, b - d} \, = \, \frac{[\, w - c^u \,]^2\, [\, b - d \,]}{2\, b - d} + \frac{[\, w - c^u \,]}{[\, 4\, b^2 - d^2 \,]^2} \{\, -d^2\, A_1$$

$$- [\, w - c^u \,]\, \left[4\, b^2 (\, 2\, b^2 - d^2\,)(f_{11} - \frac{d}{b}\, f_{12}) - d^2 (2\, b + d)(b - d) \right] \}. \qquad (111)$$

Differentiating (109) also provides:

$$\frac{\partial \Pi^u(\cdot)}{\partial f_{12}} \, = \, -\frac{d\, [\, w - c^u \,]^2\, [\, b - d \,]}{2\, b - d} + \frac{b\, [\, w - c^u \,]}{[\, 4\, b^2 - d^2 \,]^2}\, [\, A_1\, \frac{d^3}{b} + (\, w - c^u\,)\, Z \,] \qquad (112)$$

where

$$Z \, = \, 2\, b^2\, d^2 - [\, 2\, b^2 - d^2 \,]\, [\, \frac{d^2}{b}\, (b + d) \,] + 4\, b\, d\, [\, 2\, b^2 - d^2 \,]\, [\, f_{11} - \frac{d}{b}\, f_{12} \,]. \qquad (113)$$

It can be verified that:

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$$2\,b^2\,d^2 - \left[\,2\,b^2 - d^2\,\right]\left[\,\frac{d^2}{b}\,(b+d)\,\right] \;=\; -\frac{d}{b}\,d^2\,\left[\,2\,b+d\,\right]\left[\,b-d\,\right]. \tag{114}$$

(111), (112), (113), and (114) imply $\frac{\partial \Pi^u(\cdot)}{\partial f_{12}} = -\,d\,E$.

Let λ_{11} and λ_{12} denote the Lagrange multipliers associated with the first and second constraints in (108), respectively. Also let $\widehat{\mathcal{L}}$ denote the Lagrangian function associated with problem [P-VIc]. Then the necessary conditions for a solution to [P-VIc] include:

$$\widehat{\mathcal{L}}_w \;=\; \frac{\partial \Pi^u(\cdot)}{\partial w} \;=\; 0\,; \tag{115}$$

$$\widehat{\mathcal{L}}_{f_{11}} \;=\; b\,E - \lambda_{11} \;\le\; 0\,; \qquad f_{11}\,[\,\mathcal{L}_{f_{11}}\,] \;=\; 0\,; \tag{116}$$

$$\widehat{\mathcal{L}}_{f_{12}} \;=\; -\,d\,E - \lambda_{12} \;\le\; 0\,; \qquad f_{12}\,[\,\mathcal{L}_{f_{12}}\,] \;=\; 0\,. \tag{117}$$

Lemma 11. $E = 0$ at the solution to [P-VIc].

<u>Proof</u>. First suppose $E > 0$. Then $f_{11} = 1$ (because $\lambda_{11} > 0$) from (116) and $f_{12} = 0$ from (117). Therefore, from (111):

$$E \;=\; \frac{[\,w - c^u\,]^2}{[\,4\,b^2 - d^2\,]^2}\,\{\,[\,b-d\,]\,[\,2\,b+d\,]4\,b^2 - 4\,b^2[\,2\,b^2 - d^2\,]\,\} - \frac{w - c^u}{[\,4\,b^2 - d^2\,]^2}\,d^2\,A_1 \;<\; 0\,.$$

The inequality here holds because $b > d$ by assumption, $A_1 > 0$ by assumption, $w - c^u > 0$ by Lemma 10, and $2\,b^2 - d^2 > [\,b-d\,]\,[\,2\,b+d\,]$. This contradiction implies $E \le 0$.

Now suppose that $E < 0$. Then $f_{11} = 0$ from (116) and $f_{12} = 1$ (because $\lambda_{12} > 0$) from (117). Therefore, from (111):

$$E \;=\; \frac{w - c^u}{-M_1\,[\,4\,b^2 - d^2\,]^2}\,\{\,d^2\,A_1\,M_1$$

$$-\;[\,M_2 - c^u\,M_1\,]\left[\,(b-d)\,(2\,b+d)4\,b^2 + 4\,b^2(\,2\,b^2 - d^2\,)\,\frac{d}{b}\,\right]\}. \tag{118}$$

(118) reflects the fact that $w - c^u = \frac{-(M_2 - c^u\,M_1)}{-M_1}$, from Lemma 9.

$w - c^u > 0$ from Lemma 10. Furthermore, $M_1 < 0$ from the proof of Lemma 10. Therefore, because $[\,4b^2 - d^2\,]^2 > 0$, the sign of equation (118) is determined by the sign of:

$$-\,[\,M_2 - c^u\,M_1\,]\left[\,(b-d)\,(2\,b+d))\,4b^2 + 4\,b^2(\,2\,b^2 - d^2\,)\,\frac{d}{b}\,\right] + d^2\,A_1\,M_1\,. \tag{119}$$

From (105):

$$-\,[\,M_2 - c^u\,M_1\,] \;=\; [\,2\,b-d\,]\,[\,2\,b+d\,]\,A_2 - A_1\left[\,-\,2\,b\,d - d^2 + d^2\left(\,f_{11} - \frac{d}{b}\,f_{12}\,\right)\,\right]$$

$$\Rightarrow \quad -\,[\,M_2 - c^u\,M_1\,]\Big|_{f_{11}=0,\,f_{12}=1} \;=\; [\,2\,b-d\,]\,[\,2\,b+d\,]\,A_2 + A_1\left[\,2\,b\,d + d^2 + \frac{d^3}{b}\,\right]. \tag{120}$$

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From (96):

$$M_1 = -4\,[\,b-d\,]\,[\,2\,b-d\,]\,[\,2\,b+d\,]^2 + \frac{2}{b}\,[\,b-d\,]\,[\,b\,f_{11} - d\,f_{12}\,]\,\big[\,(\,2\,b-d\,)\,(\,2\,b+d\,)^2\,\big]$$

$$+\, 2\,\big[\,2\,b^2 - b\,d - d^2 + 2\,b\,(\,b\,f_{11} - d\,f_{12}\,)\,\big]$$

$$\cdot\,\bigg[\,2\,b^2 - b\,d - d^2 - \frac{1}{b}\,(\,2\,b^2 - d^2\,)\,(\,b\,f_{11} - d\,f_{12}\,)\,\bigg]$$

$$\Rightarrow \quad M_1\bigg|_{f_{11}=0,\; f_{12}=1} = -2\,\big\{\,2\,[\,b-d\,]\,[\,2\,b+d\,]\,\big[\,4\,b^2 - d^2\,\big] + 4\,b^2\,\big[\,2\,b^2 - d^2\,\big]\,\frac{d}{b}$$

$$-\,[\,2\,b+d\,]^2\,[\,b-d\,]^2 - 2\,d^4\,\big\}. \tag{121}$$

(120), (121), and (119) imply:

$$-\,[\,M_2 - c^u\,M_1\,]\,\bigg[\,(\,b-d\,)\,(\,2\,b+d\,)\,4b^2 + 4\,b^2(\,2\,b^2 - d^2\,)\,\frac{d}{b}\,\bigg] + d^2 A_1\,M_1$$

$$=\,\bigg[\,(2\,b-d)(2\,b+d)A_2 + A_1(\,2\,b\,d + d^2 + \frac{d^3}{b}\,)\,\bigg]\,\bigg[\,(\,b-d\,)\,(\,2\,b+d\,)\,4\,b^2 + 4\,b^2(\,2\,b^2 - d^2\,)\,\frac{d}{b}\,\bigg]$$

$$-\,2\,d^2\,A_1\,\bigg[\,2\,(\,b-d\,)\,(\,2\,b+d\,)\,\big(\,4\,b^2 - d^2\,\big) + 4\,b^2\,\big(\,2\,b^2 - d^2\,\big)\,\frac{d}{b} - (\,2\,b+d\,)^2\,(\,b-d\,)^2 - 2\,d^2\,\bigg]$$

$$=\,A_2\,[\,2\,b-d\,][\,2\,b+d\,]\,\bigg[\,(\,b-d\,)\,(\,2\,b+d\,)\,4\,b^2 + 4\,b^2(\,2\,b^2 - d^2\,)\,\frac{d}{b}\,\bigg]$$

$$+\,4\,b^2\,d^2\,A_1\,[\,b-d\,]\,[\,2\,b+d\,]\,\bigg[\,2\,\frac{b}{d} + 1 + \frac{d}{b} - 4 + \frac{d^2}{b^2}\,\bigg]$$

$$+\,A_1\,4\,b^2\,\big[\,2\,b^2 - d^2\,\big]\,\frac{d}{b}\,\bigg[\,(\,2\,b+d\,)d + \frac{d^3}{b} - 2\,d^2\,\bigg] + 2\,d^2\,A_1\,\big[\,(2\,b+d)^2\,(\,b-d\,)^2 + 2\,d^2\,\big] \;>\; 0.$$

The inequality holds here because $A_1 > 0$ and $A_2 > 0$. Moreover, because $b > d > 0$:

$$2\,\frac{b}{d} + 1 + \frac{d}{b} + \frac{d^2}{b^2} \;>\; 4 \quad \text{and} \quad [\,2\,b+d\,]\,d + \frac{d^3}{b} - 2\,d^2 \;=\; d\,\bigg[\,2\,b - d + \frac{d^2}{b}\,\bigg] \;>\; 0.$$

Because $w - c^u > 0$ and $M_1 < 0$, $E\,|_{f_{11}=0, f_{12}=1} > 0$, which contradicts the initial assumption that $E < 0$. Therefore, $E = 0$. $\square$

Observe that because $E = 0$ from Lemma 11, [P-VIc] is characterized by two equations, (115) and (111), and three unknowns (i.e., w, f_{11}, and f_{12}). The two equations are a function of w and $(f_{11} - \frac{d}{b} f_{12})$. Consequently, f_{11} and f_{12} are not uniquely determined. Only w and $(f_{11} - \frac{d}{b} f_{12})$ are uniquely determined.

Lemma 12. *Under collusion and VI, U_1 and U_2 set:*

$$w - c^u = \frac{d^2 A_1}{4\,b^2\,[\,2\,b+d\,]\,[\,b-d\,] - 4\,b^2\,[\,2\,b^2 - d^2\,]\,\left[\,f_{11} - \frac{d}{b}f_{12}\,\right]}. \tag{122}$$

<u>Proof</u>. The proof follows from (111), because $E = 0$ (from Lemma 11). $\square$

Lemma 13. *Under collusion and VI, U1 and U2 set:*

$$\hat{f}_1 = f_{11} - \frac{d}{b}f_{12} = \frac{F_1}{F_2} > 0, \quad where \tag{123}$$

$$
\begin{aligned}
F_1 \equiv\ & 4\,b^2\,[\,4b^2 - d^2\,]\,[\,A_1 + A_2\,]\,[\,2\,b^2 - d^2 - b\,d\,] \\
& -\ 8\,b^2 A_1\,[\,2\,b+d\,]\,[\,b-d\,]\,[\,2\,b^2 - d^2 - b\,d\,] \\
& -\ d^2 A_1\,[\,b-d\,]\,[\,2\,b+d\,]\,[\,12\,b^2 + 2\,d(\,b-d\,)\,] , \quad and
\end{aligned} \tag{124}
$$

$$
\begin{aligned}
F_2 \equiv\ & 4\,b^2\,[\,4\,b^2 - d^2\,]\,[\,A_1 + A_2\,]\,[\,2\,b^2 - d^2\,] - 8\,b^2 A_1\,[\,2\,b+d\,]\,[\,b-d\,]\,[\,2\,b^2 - d^2\,] \\
& -\ 4\,b^2\,d^2 A_1\,[\,2\,b+d\,]\,[\,b-d\,] .
\end{aligned} \tag{125}
$$

<u>Proof</u>. (122) and Lemmas 9 and 12 imply:

$$\frac{-\,[\,M_2 - c^u\,M_1\,]}{-\,M_1} = \frac{d^2 A_1}{4\,b^2\,[\,2\,b+d\,]\,[\,b-d\,] - 4\,b^2\,[\,2\,b^2 - d^2\,]\,\hat{f}_1}. \tag{126}$$

(105) and tedious calculations reveal:

$$-\,[\,M_2 - c^u\,M_1\,] = [\,2\,b-d\,]\,[\,2\,b+d\,]\,[\,A_1 + A_2\,] - A_1\left[\,2\,(\,2\,b+d\,)\,(\,b-d\,) + d^2\,\hat{f}_1\,\right]. \tag{127}$$

(96) and tedious calculations reveal:

$$-\,M_1 = 2\,[\,b-d\,]\,[\,2\,b+d\,]^2\,[\,3\,b-d\,] - \hat{f}_1\left[\,2\,(\,2\,b+d\,)\,(\,b-d\,)\,4\,b^2 - 4\,b^2\,\hat{f}_1\,(\,2\,b^2 - d^2\,)\,\right]. \tag{128}$$

(126) implies:

$$-\,M_1\,d^2\,A_1 = -\,[\,M_2 - c^u\,M_1\,]\left[\,4\,b^2\,(\,2\,b+d\,)\,(\,b-d\,) - 4\,b^2\,(\,2\,b^2 - d^2\,)\,\hat{f}_1\,\right]. \tag{129}$$

(127), (128), (129), and tedious calculations reveal $\hat{f}_1 = \frac{F_1}{F_2}$.

It is apparent that $F_2 > F_1$, so $\hat{f}_1 < 1$. Also, (124) implies $F_1 > 0$. $F_2 > F_1 > 0$ implies $\hat{f}_1 > 0$. $\square$

Lemma 14. *When collusion and VI prevail, $\frac{\partial \hat{f}_1}{\partial a_1} < 0$, $\frac{\partial \hat{f}_1}{\partial a_2} > 0$; $\frac{\partial \hat{f}_1}{\partial c_1^d} > 0$; and $\frac{\partial \hat{f}_1}{\partial c_2^d} < 0$.*

<u>Proof.</u> From (142), for $x \in \{a_1, a_2, c_1^d, c_2^d\}$:

$$\frac{\partial \hat{f}_1}{\partial x} \overset{s}{=} \left[A_2\left(2b^2 - d^2\right) + A_1 b d\right]\left[\frac{\partial A_2}{\partial x} + \frac{\partial A_1}{\partial x}\left[b - d\right]\frac{d}{2b^2}\right]$$

$$- \left[A_2 + A_1\left(b - d\right)\frac{d}{2b^2}\right]\left[\frac{\partial A_2}{\partial x}\left(2b^2 - d^2\right) + \frac{\partial A_1}{\partial x}b d\right]. \tag{130}$$

From Assumptions 6 and 7:

$$\frac{\partial A_1}{\partial c_1^d} = \frac{\partial A_2}{\partial c_2^d} = -\left[2b^2 - d^2\right]; \quad \frac{\partial A_1}{\partial c_2^d} = \frac{\partial A_2}{\partial c_1^d} = b d;$$

$$\frac{\partial A_1}{\partial a_1} = \frac{\partial A_2}{\partial a_2} = 2b; \quad \text{and} \quad \frac{\partial A_1}{\partial a_2} = \frac{\partial A_2}{\partial a_1} = d. \tag{131}$$

(130) and (131) imply:

$$\frac{\partial \hat{f}_1}{\partial c_1^d} \overset{s}{=} \left[A_2\left(2b^2 - d^2\right) + A_1 b d\right]\left[b d - \left(2b^2 - d^2\right)\left(b - d\right)\frac{d}{2b^2}\right]$$

$$- \left[A_2 + A_1\left(b - d\right)\frac{d}{2b^2}\right]\left[b d\left(2b^2 - d^2\right) - \left(2b^2 - d^2\right)b d\right]$$

$$\overset{s}{=} 2b^3 - \left[2b^2 - d^2\right]\left[b - d\right] \overset{s}{=} 2b^2 + b d - d^2 > 0.$$

The proofs that $\frac{\partial \hat{f}_1}{\partial c_2^d} < 0$, $\frac{\partial \hat{f}_1}{\partial a_1} < 0$, and $\frac{\partial \hat{f}_1}{\partial a_2} > 0$ are similar, and so are omitted. □ ∎

Proof of Conclusion 10

Observe that:

$$A_2 \lesseqgtr A_1 \iff a_2 - b c_2^d + d c_1^d \lesseqgtr a_1 - b c_1^d + d c_2^d \iff \hat{Q}_2 \lesseqgtr \hat{Q}_1$$

because:

$$A_2 - A_1 = 2b\left[a_2 - a_1\right] - d\left[a_2 - a_1\right] - b d\left[c_2^d - c_1^d\right] - \left[2b^2 - d^2\right]\left[c_2^d - c_1^d\right]$$

$$\overset{s}{=} \left[a_2 - b c_2^d + d c_1^d - \left(a_1 - b c_1^d + d c_2^d\right)\right]. \tag{132}$$

Conclusion 8 and Lemma 9 imply:

$$w^{Ic} - w^{Sc} = \frac{-\left[M_2 - c^u M_1\right]}{- M_1} - \frac{a_1 + a_2 - \left[b - d\right]\left[c_1^d + c_2^d + 2 c^u\right]}{4\left[b - d\right]}$$

$$\overset{s}{=} -4\left[b - d\right]\left[M_2 - c^u M_1\right] + M_1\left[a_1 + a_2 - \left(b - d\right)\left(c_1^d + c_2^d + 2 c^u\right)\right]. \tag{133}$$

(133) reflects the fact that $M_1 < 0$, from (100).

From (127):

$$-\left[\,M_2 - c^u\,M_1\,\right] \;=\; \left[\,2\,b - d\,\right]\left[\,2\,b + d\,\right]\left[\,A_1 + A_2\,\right] - A_1\left[\,2\,(\,2\,b + d\,)\,(\,b - d\,) + d^2\,\hat{f}_1\,\right] \quad (134)$$

where $\hat{f}_1 \;=\; f_{11} - \frac{d}{b}\,f_{12}$. From (96):

$$M_1 \;=\; -4\,[\,b - d\,]\,[\,2\,b - d\,]\,[\,2\,b + d\,]^2 + 2\,\hat{f}_1\,[\,b - d\,]\,[\,2\,b - d\,]\,[\,2\,b + d\,]^2$$

$$+\; 2\,\Big[\,2\,b^2 - b\,d - d^2 + 2\,b^2\hat{f}_1\,\Big]\,\Big[\,2\,b^2 - b\,d - d^2 - (\,2\,b^2 - d^2\,)\,\hat{f}_1\,\Big]. \qquad (135)$$

(133), (134), and (135) imply:

$$w^{Ic} - w^{Sc} \;\overset{s}{=}\; 4\,[\,b - d\,]\,[\,2\,b - d\,]\,[\,2\,b + d\,]\,[\,A_1 + A_2\,]$$

$$-\;4\,[\,b - d\,]\,A_1\,\Big[\,2\,(\,2\,b + d\,)\,(\,b - d\,) + d^2\,\hat{f}_1\,\Big]$$

$$+\;\Big[\,a_1 + a_2 - (\,b - d\,)\,\big(\,c_1^d + c_2^d + 2\,c^u\,\big)\,\Big]$$

$$\cdot\;\Big\{\,-4\,[\,b - d\,]\,[\,2\,b - d\,]\,[\,2\,b + d\,]^2 + 2\,\hat{f}_1\,[\,b - d\,]\,[\,2\,b - d\,]\,[\,2\,b + d\,]^2$$

$$+\;2\,\Big[\,2\,b^2 - b\,d - d^2 + 2\,b^2\hat{f}_1\,\Big]\,\Big[\,2\,b^2 - b\,d - d^2 - (\,2\,b^2 - d^2\,)\,\hat{f}_1\,\Big]\,\Big\}. \quad (136)$$

(103) implies:

$$a_1 + a_2 - [\,b - d\,]\,\big[\,c_1^d + c_2^d + 2\,c^u\,\big] \;=\; \frac{A_1 + A_2}{2\,b + d}\,. \qquad (137)$$

(136) and (137) imply:

$$w^{Ic} - w^{Sc} \;\overset{s}{=}\; -8\,[\,b - d\,]^2\,[\,2\,b + d\,]\,A_1 - 4\,[\,b - d\,]\,A_1\,d^2\,\hat{f}_1$$

$$+\;2\,\hat{f}_1\,[\,b - d\,]\,[\,2\,b - d\,]\,[\,2\,b + d\,]\,[\,A_1 + A_2\,]$$

$$+\;2\,\Big[\,a_1 + a_2 - (\,b - d\,)\,\big(\,c_1^d + c_2^d + 2\,c^u\,\big)\,\Big]$$

$$\cdot\;\big\{\,[\,2\,b + d\,]^2\,[\,b - d\,]^2 - 4\,b^4\,\hat{f}_1^2 + [\,2\,b + d\,]\,[\,b - d\,]\,d^2\,\hat{f}_1 + 2\,b^2\,d^2\,\hat{f}_1^2\,\big\}. \quad (138)$$

(137) and (138) imply:

$$w^{Ic} - w^{Sc} \;\overset{s}{=}\; 2\,[\,b - d\,]\,[\,A_2 - A_1\,]\,\Big[\,(\,2\,b + d\,)\,(\,b - d\,) + d^2\hat{f}_1\,\Big] - 4\,[\,b - d\,]^2\,[\,2\,b + d\,]\,A_1$$

$$+\;2\,\hat{f}_1\,[\,b - d\,]\,[\,2\,b - d\,]\,[\,2\,b + d\,]\,[\,A_1 + A_2\,] - 2\,\Big[\,\frac{A_1 + A_2}{2\,b + d}\,\Big]\,2\,b^2\,\hat{f}_1^2\,\big[\,2\,b^2 - d^2\,\big] \quad (139)$$

where the last term in (139) reflects (137).

From Lemma 13, $\hat{f}_1 \;=\; \frac{F_1}{F_2}$, where:

$$F_1 = 4b^2 \left[4b^2 - d^2\right] \left[2b + d\right] \left[b - d\right] \left[A_2 + A_1\left(b - d\right)\frac{d}{2b^2}\right] \text{ and} \tag{140}$$

$$F_2 = 4b^2 \left[4b^2 - d^2\right] \left[A_2\left(2b^2 - d^2\right) + A_1 b d\right]. \tag{141}$$

(140) and (141) imply:

$$\hat{f}_1 = \frac{F_1}{F_2} = \frac{\left[2b + d\right]\left[b - d\right]\left[A_2 + A_1\left(b - d\right)\frac{d}{2b^2}\right]}{A_2\left[2b^2 - d^2\right] + A_1 b d}. \tag{142}$$

(139) and (142) imply:

$$w^{Ic} - w^{Sc} \stackrel{s}{=} 2\left[b - d\right]\left[A_2 - A_1\right]\left[\left(2b + d\right)\left(b - d\right) + d^2 \hat{f}_1\right] - 4\left[b - d\right]^2\left[2b + d\right]A_1$$

$$+ 2\left[b - d\right]^2\left[2b + d\right]\left[A_1 + A_2\right]\left[4b^2 - d^2\right]\frac{A_2 + A_1\left[b - d\right]\frac{d}{2b^2}}{A_2\left[2b^2 - d^2\right] + A_1 b d}$$

$$- 2\left[b - d\right]\left[A_1 + A_2\right]\hat{f}_1 \frac{A_2 \, 2b^2\left[2b^2 - d^2\right] + A_1\left[b - d\right]d\left[2b^2 - d^2\right]}{A_2\left[2b^2 - d^2\right] + A_1 b d}. \tag{143}$$

(142) and (143) imply:

$$w^{Ic} - w^{Sc} \stackrel{s}{=} 2\left[b - d\right]\left[A_2 - A_1\right]\left[\left(2b + d\right)\left(b - d\right) + d^2 \hat{f}_1\right] + 2\left[b - d\right]^2\left[2b + d\right]\left[A_2 - A_1\right]$$

$$+ \frac{2\left[b - d\right]^2\left[2b + d\right]\left[A_1 + A_2\right]\left[A_2 - A_1\right]d^4}{2b^2\left[A_2\left(2b^2 - d^2\right) + A_1 b d\right]^2}\left[A_1\left(2b^2 - d^2\right) + A_1 b d\right]$$

$$\stackrel{\leq}{\gtrless} 0 \iff A_2 \stackrel{\leq}{\gtrless} A_1. \tag{144}$$

(144) holds because $A_1 > 0$ and $A_2 > 0$ by assumption, and $\hat{f}_1 > 0$ from Lemma 13. The conclusion in the lemma follows directly from (132) and (144). ∎

Proof of Proposition 4

(86) and Conclusion 7 are readily employed to prove that under collusion and VS, Di's equilibrium profit is (for $i = 1, 2$):

$$\Pi_i^{Sc*} = \frac{Q_i^2}{b} = \frac{b\left[3A_i - A_j\right]^2}{16\left[4b^2 - d^2\right]^2}. \tag{145}$$

D2's equilibrium profit under vertical separation and collusion is:

$$\Pi_2^{Sc} = \frac{b}{\left[4b^2 - d^2\right]^2}\left\{A_2 - \left[2b + d\right]\left[b - d\right]\left[w^{Sc} - c^u\right]\right\}^2. \tag{146}$$

From Conclusion 8:

$$\Pi_2^{Ic} = \frac{b}{\left[4b^2 - d^2\right]^2}\left\{A_2 - \left[w^{Ic} - c^u\right]\left[\left(2b + d\right)\left(b - d\right) + d b\left(f_{11} - \frac{d}{b}f_{12}\right)\right]\right\}^2. \tag{147}$$

$\Pi_2^{S*} = \Pi_2^{I*}$, from Conclusion 7. Therefore, (146) and (147) imply:

$$\Delta_{L_2}^l = \Pi_2^{Sc} - \Pi_2^{Ic} \overset{s}{=} \{ A_2 - [2b+d][b-d][w^{Sc} - c^u] \}^2$$

$$- \{ A_2 - [w^{Ic} - c^u] \left[(2b+d)(b-d) + db\left(f_{11} - \frac{d}{b}f_{12} \right) \right] \}^2. \quad (148)$$

(86) implies that each of the terms in $\{\cdot\}$ brackets is positive because $D1$ and $D2$ both serve customers in equilibrium, by assumption. Therefore, (148) implies:

$$\Delta_{L_2}^l \overset{s}{=} A_2 - [2b+d][b-d][w^{Sc} - c^u]$$

$$- \{ A_2 - [w^{Ic} - c^u] \left[(2b+d)(b-d) + db(f_{11} - \frac{d}{b}f_{12}) \right] \}$$

$$= [2b+d][b-d][w^{Ic} - w^{Sc}] + db[w^{Ic} - c^u]\left[f_{11} - \frac{d}{b}f_{12} \right] > 0. \quad (149)$$

The inequality in (149) holds when $A_2 \geq A_1$ because $w^{Ic} - w^{Sc} \geq 0$ when $A_2 \geq A_1$ from Lemma 10, $f_{11} - \frac{d}{b}f_{12} > 0$ from Lemma 13, and $w^{Ic} - c^u > 0$ from Lemma 10. Finally, observe from (132) that $A_2 \geq A_1 \Leftrightarrow \widehat{Q}_2 \geq \widehat{Q}_1$. $\blacksquare$

Proof of Proposition 5

From (145):

$$\lim_{d \to b} \Pi_2^{Sc} = \frac{b}{16[3b^2]^2}[3A_2^{d \to b} - A_1^{d \to b}]^2 = \frac{1}{9b^3}\left[\frac{3}{4}A_2^{d \to b} - \frac{1}{4}A_1^{d \to b} \right]^2 \quad (150)$$

where, from Assumptions 6 and 7:

$$A_i^{d \to b} \equiv \lim_{d \to b} A_i = b\left[2a_i + a_j + b\left(c_j^d - c_i^d \right) \right] \quad \text{for} \quad i = 1, 2.$$

Lemma 9 and l'Hopital's rule imply:

$$\lim_{d \to b} \left([w^{Ic} - c^u]\left[(2b+d)(b-d) + db\,\hat{f}_1 \right] \right)$$

$$= \lim_{d \to b} \frac{-[M_2 - c^u M_1]\left[(2b+d)(b-d) + db\,\hat{f}_1 \right]}{-M_1} = \frac{1}{2}A_2^{d \to b} + \frac{1}{4}A_1^{d \to b}. \quad (151)$$

(147) and (151) imply:

$$\lim_{d \to b} \Pi_2^{Ic} = \frac{b}{[3b^2]^2}\left[A_2^{d \to b} - \left(\frac{1}{2}A_2^{d \to b} + \frac{1}{4}A_1^{d \to b} \right) \right]^2 = \frac{1}{9b^3}\left[\frac{2}{4}A_2^{d \to b} - \frac{1}{4}A_1^{d \to b} \right]^2. \quad (152)$$

In the absence of collusion, $\Pi_2^I = \Pi_2^S$. Therefore, (150) and (152) imply:

$$\lim_{d \to b} \Delta_{L2}^l = \lim_{d \to b} \left(\Pi_2^{Sc} - \Pi_2^{Ic} \right) \overset{s}{=} \left[\frac{3}{4}A_2^{d \to b} - \frac{1}{4}A_1^{d \to b} \right]^2 - \left[\frac{2}{4}A_2^{d \to b} - \frac{1}{4}A_1^{d \to b} \right]^2. \quad (153)$$

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Each of the terms in square brackets in (153) is positive, given the maintained assumption that $D1$ and $D2$ both serve customers in equilibrium. Therefore, (153) implies:

$$\lim_{d \to b} \Delta_{L2}^l \overset{s}{=} \frac{3}{4} A_2^{d \to b} - \frac{1}{4} A_1^{d \to b} - \frac{2}{4} A_2^{d \to b} + \frac{1}{4} A_1^{d \to b} = \frac{1}{4} A_2^{d \to b} > 0. \quad \blacksquare$$

Proof of Proposition 6

Lemma 10 implies that $[2b+d][b-d][w^{Ic} - w^{Sc}] < 0$ when $A_2 < A_1$. In addition, Lemmas 10 and 13 imply $d\,b\,[w^{Ic} - c^u]\,[f_{11} - \frac{d}{b} f_{12}] > 0$. Furthermore:

$$\lim_{d \to 0} d\,b\,[w^{Ic} - c^u]\left[f_{11} - \frac{d}{b} f_{12}\right] = 0.$$

Consequently, when $A_2 < A_1$, there always exists a value of d sufficiently close to zero for which

$$[2b+d][b-d][w^{Ic} - w^{Sc}] + d\,b\,[w^{Ic} - c^u]\left[f_{11} - \frac{d}{b} f_{12}\right] < 0.$$

Therefore, (149) implies that $\Delta_{L2}^l < 0$ under the specified conditions. Consequently, the conclusion in the lemma follows from (132). $\blacksquare$

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