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Equilibrium Prices in Markets With Nonconvexities

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Abstract

This paper addresses the existence of market clearing prices and the economic interpretation of strong duality for integer programs in the economic analysis of markets with nonconvexities (indivisibilities). Electric power markets in which nonconvexities arise from the operating characteristics of generators motivate our analysis; however, the results presented here are general and can be applied to other markets in which nonconvexities are important. We show that the optimal solution to a linear program that solves the mixed integer program has dual variables that: (1) have the traditional economic interpretation as prices; (2) explicitly price integral activities; and (3) support an equilibrium in the presence of nonconvexities. We then show how this methodology can be used to interpret the solutions to nonconvex problems such as the problem discussed by Scarf (1994).

I. Introduction

Scarf (1990, 1994) describes most markets in today's advanced economies as having considerable indivisibilities. For example, firms must make discrete decisions on whether to invest in a new project or not, or to start-up a production process or not. These indivisibilities, which cause nonconvexities in the cost function, have important implications for markets and market clearing prices. It is widely believed that in the presence of nonconvexities, it is not possible to guarantee the existence of linear prices that will allow the market to clear.

Unfortunately, economists have largely avoided the modeling of nonconvexities such as discrete choices and economies of scale because of mathematical and analytical intractability. [1] Furthermore, in the face of nonconvexities, linear commodity prices in general will result in either a situation of excess supply or excess demand, and the market will not clear.[2] Consequently, economists have used more convenient and tractable linear or convex nonlinear optimization models to represent profit maximization problems for producers and utility maximization problems for consumers. Such optimization problems assume desirable properties such as the continuity (along with linearity or concavity) of the objective function to be maximized, and the convexity of the feasible region defined by the constraint set.[3] Hence, an equilibrium in such a market yields a linear price (or vector of linear prices) and quantity (vector of quantities) such that all economic agents maximize their objectives and the markets clear (the quantity supplied equals the quantity demanded for each commodity priced). Conceptually, a linear price vector arises out of the application of the Separating Hyperplane Theorem. [4] Moreover, such assumptions about the objective function and constraint sets allow economists to prove the existence of market clearing prices using fixed-point arguments. Computationally, if the market equilibium problem is solved by Samuelson's (1952) principle, the equilibrium prices for such markets are simply the dual variables (shadow prices or LaGrange multipliers) for the market clearing constraints for the goods in question.

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1 For example, standard graduate texts in microeconomics such as Kreps (1990) and Varian (1992) note that assuming away nonconvexities is unrealistic, but they proceed with the standard assumptions without addressing the issue further. Moreover, standard mathematical references used by economists such as Chiang (1984) and Takayama (1985) do not mention integer programming as a technique for solving optimization problems with nonconvexities.

2 As a simple example in which nonconvexities prevent a market from clearing, consider a market in which all firms have the same cost and entry is free. Each firm must incur a fixed cost of one to produce any positive amount of a good in the range (0,1]; in that range, marginal cost is zero. If the market demand curve is $P = 2 - 0.6Q$, then there is no market equilibrium. For any price less than 1, no firm will produce and there will be a shortage. For any price strictly greater than 1, quantity supplied is infinite, and there is a surplus. Finally, for $P = 1$, quantity demanded is 1.67, but the quantity supplied will be no more than 1, because if a second firm enters, it will not earn enough revenue to cover its fixed cost. Below, we examine at length another example of this phenomenon posed by Scarf (1994).

3 As a justification for the assumption of convexity, Arrow and Hahn (1971), Mas-Colell *et al*. (1995), Takayama (1985), and Varian (1992) all argue that if agents in an economy were replicated many times over, then linear prices will support a competitive equilibrium. Arrow and Hahn show how a competitive

Such modeling assumptions have allowed economists to construct useful models of economic behavior, and over the years, conduct insightful simulation experiments with these increasingly complex models. But since the work of Gomory and Baumol (1960), analogous dual variable interpretations for mixed-integer programs have eluded economists and mathematicians.[5] The economic literature continues to reflect a belief that there is no analogous dual variable interpretation of integer constraints that yield prices with any meaning. Consequently, market models are largely unable to deal with the significant nonconvexities that actually exist in markets. In particular, many firms face discrete integer decisions. For example, decisions to invest in a new capital project or not, or to start-up a production operation or not are discrete decisions. In addition to these discrete choices, many production processes have economies of scale, a property contrary to the linearity/convexity assumption. Consequently, the nonexistence of market clearing prices can be a real problem, and some degree of centralized coordination may be required in some markets to reach the welfare maximizing solution.

An important market where such nonconvexities are significant and are a concern in constructing prices is the short-term (day- to week-ahead) electric power market. Nonconvexities include start-up and shut-down costs along with minimum output requirements (which state that if a plant is running, it must produce at least a certain amount). These lumpy costs can be a large fraction of a generator's costs, and their treatment can influence optimal operating schedules and ultimately investment. It is widely recognized that the presence of these nonconvexities generally imply that there will be no linear power prices that will support an equilibrium (e.g., Johnson *et al.*, 1997, Madrigal and Quintana, 2000; Hobbs, Rothkopf *et al.*, 2001). The resulting potential mismatch of supply and demand is of concern to engineers responsible for maintaining system balance and stability, to economists and market designers who are interested in promoting market efficiency, and to the market participants themselves who are worried about how measures taken to balance supply and demand might affect their outputs and revenues.

In this paper, we present a method to construct prices for clearing markets with such nonconvexities. The method is based on methods for solving mixed integer linear programs, more usually referred to as just mixed integer programs (MIPs). MIPs contain both continuous and discrete (binary or integer valued) decision variables, and have objective functions and constraints that are linear in the decision variables. Many nonconvexities can be modeled using integer variables in MIPs. Mixed integer programming has become widely used by engineers and management scientists for solving real nonconvex problems that arise in production and consumption (e.g., Hobbs, Rothkopf *et al.*, 2001; Subramanian *et al.*, 1994). The increased use of mixed integer programming is in large measure because advances in computational methods have made finding solutions to MIPs less problematic than it used to

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equilibrium can be approximated in this case using the convex hull of the nonconvex set of constraints in the firm's or consumer's maximization problem and show the existence of this "approximate" equilibrium.

4 For example, see Takayama (1985, pp. 39-49, 103).

5 As an example, Geoffrion and Nauss (1977) state "(integer programming) models have no shadow prices or dual variables with an interpretation comparable to that in linear programming."

be. In this paper, we present a method based on mixed integer programming for finding linear prices in the presence of nonconvexities that will support equilibrium allocations in a decentralized auction-based market.

Our method for calculating such prices is relatively straightforward. First, solve a MIP to find an optimal solution. Next, delete the integrality constraints and insert the solution values of the integer variables as equality constraints (cuts) into the resulting linear program (LP). Solve the LP to find the associated dual variables on the market clearing conditions and new equality constraints. These shadow prices then can be used as prices to support an equilibrium. Such a method might be used, for instance, by the administrator of an auction market. As an example, "independent system operators" process bids and calculate prices in several short-term power markets now operating in the U.S. and elsewhere; many of these operators explicitly consider non-convexities in bids. Also, large consumers of a commodity might use this approach to design contracts with suppliers characterized by non-convex costs; an example is the power supply procurement process instituted by the U.S. Public Utilities Regulatory Policy Act of 1978 (Kahn *et al.*, 1990).

The paper proceeds as follows. Section II reviews the relevant literature. Then in Section III, we define a linear program that solves mixed-integer programs and discuss why linear prices are not enough for an equilibrium in the face of non-convexities. We then present in Section IV an example used by Scarf (1994) to show how the market clearing prices are computed. In Section V, we provide a general formulation and general proofs of our results along with an explanation of the results. Section VI concludes and discusses applications and extensions.

II. Related Literature

The economics and management science literature has occasionally addressed the problem of finding dual price interpretations to integer programs and MIPs. The classic work in this area is Gomory and Baumol (1960). In order to find the solution to the MIP, Gomory and Baumol add additional constraints (cutting planes) to the LP, which in their case they define as linear combinations of existing constraints, until the solution to the augmented LP results in an integer solution. With this methodology, they obtain shadow prices that are non-negative, impute zero profits, and infer zero prices for activities not used to capacity.

However, the shadow prices obtained by Gomory and Baumol have some peculiar properties. The prices themselves are integer valued and can vary with the choice of additional constraints. Gomory and Baumol refer to the additional constraints needed to solve the problem as "artificial", and they refer to the shadow prices on the additional constraints as "artificial capacity prices" or as the "opportunity costs of the indivisibilities." Moreover, they observe that constraints in the non-integer solution that have positive prices may have zero prices in the integer solution. For example, a warehouse may have a capacity of, say, 3.4 units, but the units only come in integer values. In this case, the capacity constraint may be binding (by making 4 units infeasible), but there is still positive slack. In an economic sense, there should be a positive price associated with this constraint.

In an attempt to deal with these peculiarities, Gomory and Baumol attempt to impute the prices from the "artificial" constraints back into the original constraints to get prices. These recomputed prices have the property that they will yield zero profit and any good with a zero price is truly a free good in an economic sense. Unfortunately, the recomputed prices may not price at zero all free goods.

The Gomory and Baumol prices also have some welfare implications. First, competitive output combinations arising from these prices will be efficient. However, Gomory and Baumol go on to state:

> "Unlike the ordinary linear programming case, however, not every efficient output can be achieved by simple centralized pricing decisions or by competitive market pricing processes. Moreover, it is possible in the integer programming case that there exists no hyperplane which separates the feasible lattice points from those which are preferred to or indifferent with the optimal lattice point. In other words, there may exist no set of prices which simultaneously makes the optimal point, *Q*, the most profitable among those that can be produced and the cheapest among those that consumers consider to be at least as good as *Q*. That is, at any set of prices either producers will try to make, or consumers will demand, some other output combination" (p. 537).

It is important to note here that Gomory and Baumol are searching for linear, uniform prices. They do admit that there are decentralized discriminatory prices that would lead to all efficient allocations, but unfortunately do not proceed further with this line of inquiry.

Scarf (1990, 1994) describes the simplex algorithm for solving LPs as being analogous to the economic institution of competitive markets, specifically a Walrasian auction. The similarities are that in a Walrasian auction, the auctioneer calls out prices until markets clear and there are zero profits, while the simplex algorithm attempts candidate solutions until no activity or slack variable can be introduced into the solution basis that improves the solution. Scarf then goes on to note that once increasing returns to scale or indivisibilities are introduced, it is difficult to draw any similar analogies between integer programming algorithms and firms or markets with such indivisibilities. Moreover, Scarf (1990) makes the following observations:

> "And, perhaps even more significant for economic theory, none of these algorithms seemed capable of being interpreted - by even the most sympathetic student - in meaningful economic terms. ... This test (for convex programs) for optimality is not available for integer programs; there simply need not be a set of prices that yields a zero profit for the activities in use at the optimal solution. ... Is its profitability at the equilibrium prices a necessary and sufficient condition for a Pareto improvement - for the possibility that everyone can be made better off using this new activity? The answer, unfortunately, is no! ... The market test fails because the firm, whose technology is based on an activity-analysis model with integral activity levels, cannot be decentralized without losing the advantages of increasing returns to scale" (p. 381-382).

In an attempt to link integer programming algorithms to economic institutions, Scarf (1990) draws the analogy of the internal structure of a large firm to an integer programming algorithm. Scarf looks at an integer programming algorithm that breaks the large integer program down into a decision tree in which smaller sub-problems can be solved in polynomial time. Scarf likens the branches of this tree to divisions of a large firm, and the nodes as managers making decisions for each of the branches below it. However, Scarf (1990) offers no method for computing prices that will help clear the market in the presence of indivisibilities, and that will provide a pricing test for Pareto improvements.

More recently, Williams (1996) discusses the mathematics of duality and its potential economic interpretations. Williams observes the same problems encountered by Gomory and Baumol in that there are often binding constraints in integer programs that have positive slack. Williams laments that this problem leads to mathematical difficulties, particularly a violation of complementarity conditions. Moreover, Williams contends that the dual prices found by Gomory and Baumol do not provide a proof of optimality (equality of primal and dual objective functions). Williams then proposes a dual as a more complex extension of Gomory and Baumol. Computation of the dual relies heavily on advanced integer optimization techniques, and in general it is difficult to associate any dual variables with a particular resource. Finally, the dual proposed by Williams, while providing a proof of optimality, still does not satisfy complementarity conditions. [6]

III. Prices in the LP that Solve the MIP

A mixed integer problem with *m* continuous variables and *n* integer variables (R^m x Z^n) that has a feasible and bounded optimal solution can be converted to a linear program with at most $m+n$ continuous variables (R^{m+n}) and at most *n* additional linear constraints (Gomory and Baumol, 1960). These statements can be proved by observing that an additional constraint can be defined for each integer variable setting the variable equal to its optimal value, which produces a LP that solves the MIP. [7] Thus, *n* can be thought of as the maximum number of additional degrees of freedom needed to price the output or the maximum additional dimensions needed for the space where the separating hyperplane or linear support function exists. In R^m, the support function is nonconvex and poorly behaved (Gould 1971). In R^{m+n}, there is always a separating hyperplane.

6 Other operations researchers have also attempted to define interpretable and computable duals/shadow prices/price functions for integer programs (Wolsey, 1981). For instance, Crema (1995) defines a shadow price based on the average incremental contribution of a resource, while Williams (1989) defines a marginal value as the directional partial derivative of the optimal objective function value with respect to perturbations in the right hand side.

7 It is worth noting here that simply solving the integer program and inserting the optimal values as equality constraints is not what Gomory and Baumol had in mind. They were primarily concerned with using cutting planes to find the solution to the integer program. As the reader will see below, we separate the issues of finding the optimal solution and identifying cuts whose duals can be interpreted as prices.

The next challenge is to find an economic interpretation of the linear prices in R^{m+n}. For convex problems there is a commodity vector for which there is a corresponding price vector. In the fixed charges example of Gomory and Baumol (1960, pp. 538-540), they assume that the additional dimensions are artificial and not meaningful. However, we believe the additional dimensions required for integer problems can be usefully viewed as additional commodities. One can think of the sub-optimality associated with integral activities and linear prices as a misspecification of the commodity space. If start-ups, or any other integral activity, are necessary for production, the auctioneer can consider these activities as separate commodities complementary to the commodity production activities that can therefore be priced as well.

Alternatively, one could think of an integral activity such as start-up much like an externality. It is well known that if externalities are present, market activities will result in sub-optimal solutions unless the externality is explicitly priced which requires placing the good (or bad) externality in the commodity space. In a linear program augmented with cutting planes that solves the integer program, one of the cuts could be a "start-up" constraint forcing the start-up variable to equal its optimal value. As we will see, the shadow price for "start-up" is the payment necessary to motivate the firm to start-up the unit or branch operation.

IV. Scarf's Example

As an example of a market with nonconvexities that lacks a market clearing price for the commodity, consider the problem put forth by Scarf (1994). He postulates two types of firms, each with significant fixed costs and relatively small marginal costs (Table 1). The objective of the problem is to minimize the total cost of satisfying a fixed level of demand.

Table 1. Production Characteristics: Smokestack versus High Tech (from Scarf, 1994)

Characteristic	**Smokestack (Type 1 Unit)**	**High Tech (Type 2 Unit)**
Capacity	16	7
Construction Cost	53	30
Marginal Cost	3	2
Average Cost at Capacity	6.3125	6.2857
Total Cost at Capacity	101	44

Suppose that we were to attempt to satisfy a fixed demand of 61 units. The optimal solution to this problem would be to build 3 Smokestack units and 2 High Tech units with each running at full capacity except for the last Smokestack unit that only produces 15 units. What should the prices be? In the context of linear prices, candidate prices might include the marginal production costs of each type and the average costs at full capacity of each type. Yet if price equaled either of the marginal costs (2 or 3), neither type of technology would want to produce. Each type would incur losses, so it is profit maximizing at those prices to neither build nor produce. But on the other hand, if the price equaled the average cost of the Smokestack technology at capacity (6.3125), then two Smokestack type units would be making zero profits, and the third Smokestack unit would be operating at a loss. At this price, the High Tech types would be making positive profits, and an infinite number of this type would want to enter the market. This cannot be an equilibrium since there would be excess supply at this price. The only other serious candidate price is the average cost of the High Tech type at capacity (6.2857). At this price the High Tech types would make zero profit if they operate at full capacity, but the Smokestack types would still incur losses. Therefore, no Smokestack types would wish to enter; further, if enough High Tech types enter to meet the demand of 61, the last unit would not be operating at capacity, and would be incurring losses. Thus, a price of 6.2857 cannot be an equilibrium either.

Now, consider the construction (start-up) for each type as a separate commodity so that there are now three commodities that must be priced: the final output, construction of the Smokestack type, and construction of the High Tech type. Let a price of 3, the marginal cost of the higher cost type, be the candidate price for the final output. Let a price of 53, the construction cost of the Smokestack type, be the candidate price for building the Smokestack type. Finally, let a price of 23 be the candidate price for building High Tech types. A price of 3 on the final output makes sense, in the example above, since the third Smokestack unit can produce one more unit at a marginal cost of 3 before being at capacity. At the candidate prices, all Smokestack units would receive a price of 3 for the final output that they can produce at a marginal cost of 3. Each Smokestack unit then receives a price of 53 for construction, leaving each Smokestack unit with zero profits. The High Tech units each receive a price of 3 for the final output that they can produce at a marginal cost of 2, leaving each High Tech unit with a margin of 1 per unit of output. At the candidate construction price of 23, each High Tech unit is left with precisely zero profit. Note that the construction price that High Tech units receive is not equal to its actual construction costs. If the market were to naively pay them actual construction costs, the High Tech units would be making positive profits which would lead to entry of an infinite number of High Tech units and excess supply.

Thus, if start-up decisions in that example are viewed as commodities, equilibrium supporting prices can be constructed. It turns out that these prices are the dual variables for a linear program augmented by two cuts that define the number of Smokestack and High Tech units as equaling 3 and 2, respectively. In the remainder of this section, we analyze Scarf's problem further and then present the original mixed integer programming formulation along with the augmented LP that solves it.

Table 2 presents the least-cost solutions for demands ranging from 55 to 70 units in the example presented by Scarf (1994). We solved the problems using Excel® Premium Solver. We calculated market-clearing prices for these problems using the following procedure:

1. Formulate the problem as a mixed integer program and solve.
2. Find a LP that solves the MIP by adding cuts that set the integer variables to their optimal values.
3. Use the dual variables and primal quantities from the linear program to form an efficient contract.

Table 2. Cost Minimizing Choices of Plants and Output Levels (from Scarf 1994)

Demand	Number of Type 1 Units (Smokestack)	Number of Type 2 Units (High Tech)	Output of Type 1 Units	Output of Type 2 Units	Total Cost
55	3	1	48	7	347
56	0	8	0	56	352
57	1	6	15	42	362
58	1	6	16	42	365
59	2	4	31	28	375
60	2	4	32	28	378
61	3	2	47	14	388
62	3	2	48	14	391
63	0	9	0	63	396
64	4	0	64	0	404
65	1	7	16	49	409
66	2	5	31	35	419
67	2	5	32	35	422
68	3	3	47	21	432
69	3	3	48	21	435
70	0	10	0	70	440

A MIP formulation of the Scarf problem in which each firm's decision to enter is represented as an integer variable is[8]

8 The formulation shown here can take an unnecessarily long time to solve unless modern MIP software is used (Hobbs, Stewart *et al.*, 2001). An equivalent formulation that would solve more quickly on basic

Minimize: $$? _i (53z_{1i} ? 3q_{1i}) ? ? _j (30z_{2j} ? 2q_{2j})$$

subject to: $$? _i q_{1i} ? ? _j q_{2j} ? Q$$

$$?16z_{1i} ? q_{1i} ? 0 \qquad ? i$$

$$?7z_{2j} ? q_{2j} ? 0 \qquad ? j$$

$$q_1, q_2 ? 0 \qquad ? i, j$$

$$z_{1i}, z_{2j} ? \{0,1\} \qquad ? i, j,$$

MIP solvers defines z_1 and z_2 as representing the total numbers of units of types 1 and 2, respectively, and q_1 and q_2 as representing their total output.

where

z_{1i} and z_{2j}	represent the decision to start up unit i (i = 1,2,…, I) of type 1 (Smokestack) and unit j (j=1,2,…,J) of type 2 (High Tech), respectively,
q_{1i} and q_{2j}	are the quantities of output for Smokestack unit i and High Tech unit j, respectively, and
I and J	are the numbers of Smokestack and High Tech units, respectively that are considered in the MIP. These numbers should be large enough so that they do not constrain the solution; otherwise they will generate scarcity rents.

A linear program that solves the above MIP is

			Dual Variables
Minimize:	$? _i (53z_{1i} ? 3q_{1i}) ? ? _j (30z_{2j} ? 2q_{2j})$		
subject to:	$? _i q_{1i} ? ? _j q_{2j} ? Q$		y
	$?16z_{1i} ? q_{1i} ? 0$	$? i$	y_{1i}
	$z_{1i} ? z^*_{1i}$	$? i$	w_{1i}
	$?7z_{2j} ? q_{2j} ? 0$	$? j$	y_{2j}
	$z_{2j} ? z^*_{2j}$	$? j$	w_{2j}
	$q_{1i}, q_{2j} ? 0$	$? i, j,$	

where

z^*_{1i} and z^*_{2j}	are the optimal values from MIP2, and
y	is the single commodity price for all production,
y_{1i}	is the capacity dual variable for the ith Smokestack unit,
w_{1i}	is the start-up price for the ith Smokestack unit,
y_{2j}	is the capacity dual variable for jth High Tech unit, and
w_{2j}	is the start-up price for the jth High Tech unit.

Table 3 summarizes the values of the dual variables from solving the LP for each of the instances in Table 2. As we show in the next section, the dual variables for the market clearing and integer variable constraints, when used as prices, collectively can be used by a market operator (auctioneer) to define a contract that clears the market and is efficient. Each firm i of type t is paid $w_{ti}^* z_{ti}^*$ for starting up and $y^* q_{ti}^*$ in exchange for producing q_{ti}^*.[9] Negative prices are payments to the auctioneer as part of the contract. As we will also show in the next section, this contract yields nonnegative profits for each player. Further, these prices support an equilibrium. That is, under these contract offers, those producing and those not producing are both economically satisfied with the individual contract offer, in the sense that under the offered prices, no other levels of output would increase profit. Finally, the solution is efficient (in this case, least cost).

[9] In general, it is necessary to specify the quantity to be produced in the contract because price signals alone as decentralized mechanisms are not always sufficient to clear the market for either convex or nonconvex problems. This is not a characteristic of MIPs alone, but is also true for linear programs. In con-

Table 3: Dual Prices for Scarf's Problem

		Unit 1 (Smokestack)		Unit 2 (High Tech)	
Dual Price Set	**Commodity Price**	**Start-up Price**	**Capacity Price**	**Start-up Price**	**Capacity Price**
Set I[a]	3	53	0	23	-1
Set II[b]	6.3125	0	-3.3125	-.1875	-4.3125
Set III[b]	6.2857	.429	-3.2857	0	-4.2857

a. Applies to all integer demand levels from 55 to 70

b. There are alternative dual solution for demands of 55,56,58,60,62,63,64,65,67,69,70; for these solutions, each unit in the solution of both types is producing at its capacity.

In Scarf's example, the commodity price (y*) is either the variable cost of unit 1 (the highest unit marginal operating cost), the average cost of unit 1 at full output, or the average cost of unit 2 at full output (Table 3). In this case, there are often alternative optimal contracts, depending on the level of demand, as indicated in Table 3. These result from degeneracy in the primal LP, stemming from the coincidence that demand exactly equals the sum of the capacities of the units in the solution. However, each set of contract terms yields the same revenue and output result. Each also has an economic interpretation. There are three basic pairs of contracts that use the dual variables in Table 3:

Contract Pair I (using dual price set I):

1. For Smokestack units: produce q_{1i}*; get paid $3/unit of production (the highest marginal cost of a running unit); and get paid $53 to start up.
2. For High Tech units: produce q_{2i}*; get paid $3/unit of production; and get paid $23 to start up.

Contract Pair II (dual price set II):

1. For Smokestack units: produce q_{1i}*; get paid $6.3125/unit of production (average cost of unit 1 at capacity); and get paid 0 to start-up.
2. For High Tech units: produce q_{2i}*; get paid $6.3125/unit of production; and get charged $0.1875 if the unit is started up.

Contract Pair III (dual price set III):

1. For Smokestack units: produce q_{1i}*; get paid $6.2857/unit of production (average cost of unit 2 at capacity); and get paid $0.429 to start up units.

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vex optimization, only cost functions that are strictly convex at the equilibrium will, in general, allow for pure price signals in an auction context. Otherwise, quantities must be included in the auctioneer's contract offer when there are alternative optimal responses to a given price. For instance, if a supplier is on the flat part of a marginal cost curve, the auctioneer must send quantity signals in addition to price signals to obtain a feasible solution that clears the market without excess supply or demand.

2. For High Tech units: produce q_{2i}*; get paid \$6.2857/unit of production, and get paid \$0 to start up units.

Contract pair I will, for all levels of demand, clear the market. Any of the three contracts will work when both plants types run at capacity (under demands of 55, 56, 58, 60, 62-65, 67, 69, 70). However, under other demands (57, 59, 61, 66, 68), one or more Smokestack units operate at partial capacity because that type has the higher marginal cost. In those cases, Contracts II and III do not support an equilibrium, as Smokestack units not running at capacity would not recover their costs. Those units would also like to produce more at these prices since their marginal cost is only \$3/unit when running at less than capacity.

In this example, it turns out that all units producing are offered a contract that pays exactly their costs. The start-up payment is the difference between total cost and the commodity revenues. But in general, profits (scarcity rents) can be positive if, for instance, some firms possess uniquely low cost technologies. In the Scarf example, however, there are an infinite number of potential entrants with costs identical to firms in the solution, so for an equilibrium to occur, no firms can be earning positive rents.

These linear price contracts can be viewed as being analogous to multi-part prices for commodities. For instance, a start-up payment can be viewed as being similar to a demand or customer charge in utility pricing. In general, it is well recognized that multi-part pricing is necessary for efficient pricing in the presence of nonconvex costs. As an example, in the presence of demand with non-zero elasticity, the best one-part prices are Ramsey (1927) prices and are "second best" when compared with efficient multi-part prices. Optimal multi-part prices derived in the above manner are similar to a solution to a cooperative bargaining problem (Luce and Raiffa 1957) and to optimal multi-part pricing for natural monopolies such as demand and commodity charges in regulatory contracts (Brown and Sibley 1986). It is already known that lump-sum payments or multi-part prices are necessary for efficient pricing of a single good with nonconvex costs (see *ibid.* and Sharkey (1982)). It then should come as no surprise that more degrees of freedom for pricing in the contract allows for greater efficiency in a market with no transaction costs. For MIPs, the pricing degrees of freedom needed are bounded by the sum of the number of explicit constraints and the number of integer variables.[10] If the buyers' problems are also MIPs, optimal contracts can be devised for both buyers and sellers. In the next section, we present our general results for all markets that can be represented by mixed integer programs.

[10] In our experience in solving electric power unit commitment models, the number of non-zero prices associated with start-up and shut-down decisions is one to two orders of magnitude smaller than the number of such variables. However, in general, the number of additional prices could in theory equal the number of integer variables.

V. General Formulation and Proofs

In this section, we present a result concerning the equivalence of a MIP and an LP augmented with certain defined cutting planes. We then define a contract that an auctioneer might offer that is efficient and that has prices that support a market clearing equilibrium. The contract can be viewed as a modified second-price auction because each participant in the auction is paid the same price for the commodity it provides, but in addition may receive payments associated with its discrete decisions, represented as integer variables. Although these results are phrased as if they apply only to formal auction markets, they are also applicable to other markets.

Consider an auction market that can be represented by a **P**rimal Mixed **I**nteger **P**rogram (PIP). The formulation below assumes that the auctioneer is buying and/or selling a set of goods, and has an objective of maximizing the value bidders receive from them. The auctioneer is simply a computer code that finds a solution to the problem: [11]

PIP Maximize: $$v_{PIP} \;=\; ?_{k} c_k x_k \;?\; ?_{k} d_k z_k$$

Subject to:

$$\begin{array}{ll} ?_{k} A_{k1} x_k \;?\; ?_{k} A_{k2} z_k \;?\; b_0 & \\ B_{k1} x_k \;?\; B_{k2} z_k \;?\; b_k & ?\, k \\ x_k \;?\; 0 & ?\, k \\ z_k \;?\; \{0,1\}^{n(k)} & ?\, k\,, \end{array}$$

where

x_k, z_k are activities or column vectors of activities for participant k in the market (k ? K),

c_k, d_k are the benefits (scalars or vectors) associated with activities of participant k (scalars or row vectors). Thus, $c_k x_k + d_k z_k$ is the benefit accruing to participant k,

A_{k1}, A_{k2}, B_{k1}, B_{k2} are matrices of constraint coefficients,

b_k represents the right hand sides of internal constraints of the market participant k (scalars or column vectors),

b_0 represents commodities to be auctioned by the auctioneer (a scalar or column vector), (In a double auction where the auctioneer just facilitates trades, $b_0 = 0$.), and

$\{0,1\}^{n(k)}$ is the set of 0-1 activity vectors of cardinality $n(k)$, and $n(k)$ is the number of rows in vector z_k.

Lower case characters represent scalars or vectors; upper case characters represent matrices; all multiplication is of compatible dimensions.

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[11] In general, these problems may be hard to solve because, with a few special exceptions, MIPs are NP-hard problems (*i.e.*, there is no algorithm whose solution time is guaranteed not to increase exponentially with the size of the problem) (*e.g.*, Johnson *et al.* (1997)).

A **P**rimal **L**inear **P**rogram that solves PIP is:

PLIP(z*) Maximize: $v_{PLIP} = ?_k c_k x_k ? ?_k d_k z_k$

subject to:

$$?_k A_{k1} x_k ? ?_k A_{k2} z_k ? b_0$$
$$B_{k1} x_k ? B_{k2} z_k ? b_k \qquad ? k$$
$$z_k ? z_k^* \qquad ? k$$
$$x_k ? 0 \qquad ? k ,$$

where z_k* represents the values of the z_k variables in an optimal solution to PIP. In general, PLIP contains more constraints than PIP; these are needed for the LP to solve the MIP and to yield strong duality. The dual of PLIP(z*) is:

DLIP(z*) Minimize: $v_{DLIP} = y_0 b_0 ? ?_k y_k b_k ? ?_k w_k z_k^*$

subject to:

$$y_0 A_{k1} ? y_k B_{k1} ? c_k \qquad ? k$$
$$y_0 A_{k2} ? y_k B_{k2} ? w_k ? d_k \qquad ? k$$
$$y_0 ? 0$$
$$y_k ? 0, \qquad ? k$$
$$w_k \text{ unrestricted} \qquad ? k ,$$

where y_0, y_k, w_k are the dual variables, either scalars or appropriately dimensioned row vectors.

Theorem 1: v_{PIP}* = v_{PLIP}* = v_{DLIP}*, where * indicates the optimal solution value for the respective problems.

Proof: v_{PIP}* = v_{PLIP}* because PLIP is PIP with the additional constraints that the integer variables are constrained to their optimal values (which then allows the integrality condition $z_k ? Z_k$ of PIP to be dropped as redundant). v_{PLIP}* = v_{DLIP}* by strong duality of linear programs. ✍

Definition 1: A **market clearing set of contracts** is a set of contracts with the following characteristics:

1. Each bidder is in equilibrium, in the sense that given:

    ?? the prices {y_0*, y_k*, w_k*} and payment function defined by the contract, and

    ?? no restrictions on x_k and z_k other than the bidder's internal constraints ($B_{k1}x_k + B_{k2}z_k ? b_k$, $z_k ? Z_k$), no bidder k would be able to increase its net benefit ($c_k x_k$ + $d_k z_k$ minus its payment) over that received if $x_k = x_k$* and $z_k = z_k$*. Thus, the prices support the equilibrium solution {x_k*,z_k*}.

2. Supply meets demand for the commodities (*i.e.*, ? $_k$ $(A_{k1}x_k* + A_{k2}z_k*)$? b_0).

Definition 2: Let contract $\mathbf{T_k}$ be a contract between the auctioneer and bidder k with the following terms:

1. Bidder k buys (or sells) $z_k=z_k*$, $x_k=x_k*$;

2. Bidder k pays an amount to the auctioneer equal to the following payment function: $y_0*(A_{k1}x_k+A_{k2}z_h) + w_k*z_k$.

where, as before, * indicates an optimal solution to PLIP or DLIP, as appropriate. This contract is a modified uniform price auction contract. Let **T** ? {$\mathbf{T_k}$ for each k ? K}.

Theorem 2: **<u>T</u>** is a market clearing set of contracts.

Proof: Let the optimal solution to PLIP(z*) be {x_k*,z_k*′} and the optimal solution to DLIP(z*) be {y_0*, y_k*, w_k*}. (We use the notation z_k*′ to distinguish the optimal value of the variable z_k from the fixed right hand side of the constraint z_k = z_k* in PLIP(z*).) The Karesh- Kuhn-Tucker conditions for optimality of these problems are:[12]

$0 ? (y_0*A_{k1} + y_k*B_{k1} - c_k) ? x_k* ? 0$? k ,
$0 ? (y_0*A_{k2} + y_k*B_{k2} + w_k* - d_k) ? z_k* ? 0$? k ,
$0 ? y_0*? (?_k A_{k1}x_k* + ?_k A_{k2}z_k*? - b_0) ? 0$
$0 ? y_k* ? (B_{k1}x_k* + B_{k2}z_k*? - b_k) ? 0$? k ,
$w_k*(z_k*?- z_k*) = 0$? k ,

Now consider the following problem. Say that when the auctioneer defines **<u>T</u>**, each participant k is offered prices {y_0*, w_k*} (term 2 of the contract), but their primal variables are unconstrained (term 1 is not enforced). Then each participant k will solve the following MIP of maximizing its benefits minus payment, subject to its internal constraints:

$\mathbf{PIP_k}$ Maximize: $v_{PIPk} = (c_k x_k + d_k z_k) - y_0*(A_{k1}x_k+A_{k2}z_k) - w_k*z_k$

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[12] Note that "0 ? $f(x)$? x ? 0" is shorthand for the following complementarity condition for a scalar or column vector x and a function $f(x)$ of the same dimension as x:

$0 ? f(x);\ x ? 0;\ f(x)^T x = 0.$

$$\text{subject to:} \quad B_{k1}x_k + B_{k2}z_k \ ? \ b_k \qquad ? \ k$$
$$x_k \ ? \ 0, \qquad ? \ k$$
$$z_k \ ? \ Z^k \qquad ? \ k \quad .$$

Let $v_{PIPk}*$ be the value of the objective of PIP$_k$ at $\{z_k*, x_k*\}$. We can show that $v_{PIPk}* = y_k*b_k$ as follows. Insert $\{z_k*, x_k*\}$ into the objective of PIP$_k$, and then add the term $y_k*(B_{k1}x_k* + B_{k2}z_k* - b_k)$ to the objective (which is permissible, since by the complementary slackness conditions given above, that term equals zero), and then cancel terms:

$$\begin{aligned} v_{PIPk}* &= (c_k x_k* + d_k z_k*) - y_0*(A_{k1}x_k* + A_{k2}z_k*) - w_k*z_k - y_k*(B_{k1}x_k* + B_{k2}z_k* - b_k) \\ &= (c_k - y_0*A_{k1} - y_k*B_{k1})x_k* + (d_k - y_0*A_{k2} - y_k*B_{k2}z_k* - w_k*)z_k* + y_k*b_k \\ &= y_k*b_k \end{aligned}$$

The third equality follows because the first and second terms in the second equality each equal zero by the complementary slackness conditions given earlier.[13]

Now, let the optimal solution to PIP$_k$ be $v_{PIPk}**$. If is $v_{PIPk}**$ is less than or equal to $v_{PIPk}*$ for each k, then the contract **T** is market clearing for the reasons below:

? no participant can obtain a feasible $\{x_k, z_k\}$ giving a greater profit in PIP$_k$ than $\{x_k*, z_k*\}$, and

? as $\{x_k*, z_k*\}$ by definition solves PLIP, they also satisfy the market clearing condition $?_k(A_{k1}x_k + A_{k2}z_k) \ ? \ b_0$.

The last thing that must be shown is that $v_{PIPk}** \ ? \ v_{PIPk}*$ is indeed true. To demonstrate this, rearrange the terms of $v_{PIPk}**$ to yield the following:

$$v_{PIPk}** = \text{Maximize } [(c_k - y_0*A_{k1})x_k + (d_k - y_0*A_{k2} - w_k*)z_k]$$
$$\text{subject to:} \quad B_{k1}x_k + B_{k2}z_k \ ? \ b_k \qquad ? \ k$$
$$x_k \ ? \ 0 \qquad ? \ k$$
$$z_k \ ? \ Z^k \qquad ? \ k \ .$$

Now let $\{x_k**, z_k**\}$ be the optimal solution for PIP$_k$. As a result, $v_{PIPk}** = [(c_k - y_0*A_{k1})x_k** + (d_k - y_0*A_{k2} - w_k*)z_k**]$. Now add the following nonnegative term to $v_{PIPk}**$:

$$-y_k*(B_{k1}x_k** + B_{k2}z_k** - b_k).$$

This term is nonnegative because $y_k* \ ? \ 0$ (see the PLIP complementary slackness conditions, above) and $B_{k1}x_k** + B_{k2}z_k** \ ? \ b_k$ (by the definition of PIP$_k$). As a result:

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13 Note that since both y_k and b_k are nonnegative, v_{PIPk} too is nonnegative, and all bidders must earn nonnegative (and perhaps positive) profits under contract **T**.

v_{PIPk}** ? $[(c_k - y_0{}^*A_{k1})x_k{}^{**} + (d_k - y_0{}^*A_{k2} - w_k{}^*)z_k{}^{**}] - y_k{}^*(B_{k1}x_k{}^{**} + B_{k2}z_k{}^{**} - b_k)$
$= [(c_k - y_0{}^*A_{k1} - y_k{}^*B_{k1})x_k{}^{**} + (d_k - y_0{}^*A_{k2} - y_k{}^*B_{k2} - w_k{}^*)z_k{}^{**}] + y_k{}^*b_k$
? $y_k{}^*b_k = v_{PIPk}{}^*$.

The last inequality results from noting that:

1. $(c_k - y_0{}^*A_{k1} - y_k{}^*B_{k1})x_k{}^{**}$? 0, because $(c_k - y_0{}^*A_{k1} - y_k{}^*B_{k1})$? 0 (from the definition of DLIP, above) and $x_k{}^{**}$? 0.
2. $(d_k - y_0{}^*A_{k2} - y_k{}^*B_{k2} - w_k{}^*)z_k{}^{**}$? 0, because $(d_k - y_0{}^*A_{k2} - y_k{}^*B_{k2} - w_k{}^*)$? 0 (again from DLIP) and $z_k{}^{**}$? 0.

Consequently, we have shown that $v_{PIPk}{}^{**}$? $v_{PIPk}{}^*$; *i.e.*, no participant k can obtain a feasible solution giving a greater profit for PIP_k than the auctioneer's solution $\{x_k{}^*, z_k{}^*\}$. ✍

Theorem 2 shows that the dual solution to the constructed linear program PLIP that solves the mixed integer program PIP can be used to form a contract for all bidders. The contract uses the primal quantities and the dual variables for the commodity and integer constraints as prices. In this modified uniform price auction, bidders may make or receive payments associated with their lumpy decisions. In contrast, in a traditional uniform price auction, a bidder is just paid the marginal price (the dual variable on the market clearing constraint) for its output and ignores the dual variable on the individual capacity constraint.

Theorem 2 in the context of Scarf's problem provides other insights. For instance, there are some levels of quantity demanded for which both Smokestack and High Tech units are "inframarginal" in the sense that their marginal costs are less than the commodity price and all are operating at capacity (see Table 3). In the linear program, the resulting scarcity rents appear as positive dual variables on binding upper bounds of activities. And in some of these instances, High Tech units have negative start-up payments, indicating that scarcity rents exceed start-up costs. Such negative start-up payments can occur in order to dissuade uneconomic entry. For instance, if any unit of a widely available type is collecting scarcity rents, then an infinite number of those units will wish to enter and the market will not clear. However, in auctions where entry cannot occur instantaneously (e.g., daily power markets), then rents can be earned by units under a **T** contract even when, in the long run, the technology is widely available.

Examples of auctions formulated in a manner similar to, and yielding linear prices similar to **T** can be found in the New York Independent System Operator (NYISO) and the Pennsylvania-New Jersey-Maryland Interconnection (PJM) electric energy markets. In these markets, the market operator explicitly asks generators to bid costs associated with non-convexities (start-up and minimum load). For example, suppose a generating unit is started up in order to meet energy or reserve margin (spare capacity) constraints for the entire system. If the revenues from sales of energy and reserves fail to cover those costs, then the auctioneer provides a lump sum payment to the generator to make up the difference. On the other hand, if a generating unit's scarcity rents associated with binding internal capacity constraints are greater than start-up costs, then the generating units are allowed to keep the rents, effectively ignoring the dual variable on the start-up constraint.

Theorem 3: If each participant k submits a bid reflecting its true valuations ($c_k x_k + d_k z_k$) and true constraints ($B_{k1} x_k + B_{k2} z_k$? b_k; x_k ? 0; z_k ? Z^k), an auction defined as follows maximizes net social benefits (?$_k$ [$c_k x_k + d_k z_k$]) and is market clearing:

1. The auctioneer first solves problem PIP, yielding primal solution {x_k*,z_k*};

2. The auctioneer determines prices {y_0*, w_k*} by solving problem PLIP(z*); and

3. The auctioneer offers contract **T**.

Proof: By definition, the solution {x_k*, z_k*} of PIP maximizes net social benefits and satisfies the second condition of market clearing (?$_k$ [$A_{k1} x_k + A_{k2} z_k$] ? b_0). The only remaining condition is whether the prices from PLIP(z*) support this solution. Theorems 2 demonstrates this for the payment schemes in **T**. ✍

Theorem 3 is an extension, to auctions with nonconvexities, of the Fundamental Theorem of Welfare Economics, which states that a competitive equilibrium is Pareto Optimal.

VI. Conclusions, Applications, and Extensions

This paperhas addressed a problem that has troubled the economic analysis of markets with non-convexities: the existence of market clearing prices. Given the presence of non-convexities in emerging electricity auctions , this problem is of practical as well as theoretical interest. We have shown that the optimal solution to a linear program that solves the mixed integer program representing the non-convex problem yields shadow prices (dual variables) that have an economic interpretation, and can be used to design an optimal contract with prices that support an equilibrium in a decentralized market. The contract defined by **T** provides an answer to Scarf's (1994) search for a set of prices in the presence of non-convexities that yield zero profits for all activities in the optimal solution. A modified implementation of **T**, in contrast, allows positive profits for activities in the solution, and bears strong resemblance to actual auction mechanisms used in some power markets. These results hold for any market that can be represented by mixed integer program.

Given recent advances in computational technology and integer programming algorithms, finding the prices necessary to define these contracts is practical. Roughly speaking, MIPs today take on average about the same or less time (wall clock) that linear programs of similar size took to solve in the 1960s (Ceria, 2001; Hobbs, Stewart *et al.*, 2001).[14] Therefore, the results presented here are not relevant just to toy problems. In particular, applying this approach to electric generating unit commitment auctions could be a significant step forward. As pointed out in Section I, the generating unit commitment problem for day-ahead energy markets is inherently non-convex due to the discrete choice of whether or not to start-up a unit and minimum run levels. The problem facing the market operator is that in general *linear prices in energy alone* do not exist to support an equilibrium. As mentioned above, new and evolving electricity auction markets like PJM and NYISO have implemented market and pricing mechanisms similar to the one discussed in this paper. Yet, there are other electric power auction mechanisms such as those in California and New England that do not accept non-convex bid functions, leaving it up to the individual generators to average in costs related to start-ups in their energy bids.

Now that we can define an equilibrium in markets with non-convexities, there are many questions that can be examined. First, Scarf's (1994) search for price based tests for Pareto improving entry can be re-examined. For example, if any potential activity can make a positive profit under the prices and quantities specified in contract **T,** then it should be included in the solution. Future work should investigate the definition and properties of such tests. However, such tests are unlikely to be both necessary and sufficient for evaluating the profitability of such activities in non-convex problems; in general, there may be some activities that fail those tests, yet their inclusion would still increase profit. A definitive test is to include the activity as a decision variable in the MIP and resolve the model. Fortunately, improved capabilities in mixed integer programming make that a more practical approach than it once was.

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[14] With respect to computational times, the theoretical upper bounds on calculations have usually been much greater than the actual solution times for applications. There are several possible explanations for this discrepancy. First, it may be that actual applications seldom encounter pathological problems. Second, the difficult to solve problems are shelved. Third, the problems can often be reformulated to remove many pathologies.

Second, much has been made in the electricity industry about the possibilities for strategic bidding behavior to manipulate prices (e.g., Borenstein and Bushnell, 1999). Adding another bidding parameter, such as an integral activity like start-up costs, gives generators another degree of freedom that they can manipulate strategically.[15] An examination of the possibilities of a whether a greater exercise of market power, and hence higher market power rents, are possible in the auction market proposed in this paper versus simple auction in which non-convexities are ignored is required to address the above issue. In the context of such a study, issues like what bid parameters (integral or continuous) should be bid strategically to maximize profit, and what kind of activity rules hinder or help such strategic behavior. Moreover, the auction pricing mechanism proposed in this paper could be compared a first-price and Vickery-Clarke-Groves auction mechanisms.[16]

Third, the efficiency of the auction pricing mechanism proposed here can be compared to the efficiency of simple auctions that ignore non-convexities. In particular, an efficiency comparison of the MIP based auction to a simple (commodity only) one-time auction and to a simple (commodity only) repeated auction would be of interest. In the context of electricity markets, the above comparison may have interesting implications. In terms of overall costs, the cost of start-up/minimum-load payments may be small relative to energy costs. In PJM, start-up/minimum-load payments per MWh of load are about 100 times less (roughly \$0.30/MWh vs. \$30/MWh).[17] While the overall cost impact of non-convex decisions may be small, these costs can be a significant portion of generating total generating costs to generators serving peak load or reliability functions. Moreover, without this mechanism, generators may receive physically infeasible dispatch orders.

Finally, our results say nothing about the uniqueness of equilibrium prices. In fact, as can be seen in Scarf's example in Section IV, there can be multiple equilibria.[18] Alternative equilibrium prices might lead to different distributions of surplus for market participants under contract **T**. Given that there is a lot of money at stake in the new electricity markets, where the bidding of non-convex costs is already taking place, an examination of the distributional consequences of alternative equilibria is of keen interest to these market participants.

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[15] One intuitive observation can be made about strategic behavior. In the context of a sellers auction where the technologies are widely available and entry is instantaneous (as in the Scarf example in Section IV), even if the participants are not constrained to bid costs, a MIP auction solution produces a Nash equilibrium in which all generators bid their costs. The reason is that if anyone bids above its costs it would be immediately undercut by an entrant with the same costs. However, while this may be a good point of departure, the reality of market power in markets with integral activities is much different.

[16] See Hobbs *et al.* (2000) for a start at this.

[17] Personal Communication, Andy Ott, PJM

[18] Admittedly, in simple examples degeneracy of the augmented LP can be a problem, leading to multiple dual solutions. However, in larger more complex problems, it is not entirely clear how big a problem a multiplicity of solutions will be. Here, for instance, it turns out that the multiple equilibria in Scarf's prob-

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lem result from the assumed equality of costs of different suppliers. In reality, costs and bids are seldom exactly equal.

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