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Note

ESTIMATING GEOGRAPHIC CUSTOMER DENSITIES USING KERNEL DENSITY ESTIMATION

NAVEEN DONTHU AND ROLAND T. RUST
Georgia Institute of Technology
University of Texas at Austin

This paper shows how kernel density estimation may be used to estimate flexibly the geographic distribution of customers in a market. In addition it shows how a density-based product positioning methodology may be applied to site selection, using the estimated geographic customer density to help locate a new or (relocated) store or distribution center. This application provides a conceptual basis for more complicated site selection and spatial demand models which might involve several predictor variables.
(Market Area Analysis; Retailing; Kernel Density Estimation; Nonparametric Density Estimation; Site Selection)

1. Introduction

Retailers and service marketers are often faced with questions which depend upon the geographic distribution of customers in a market. Some of these questions are:

1. Where should a new store (or service facility) be located, in order to maximize market share (or utilization)?
2. Where should a store (or service facility) be relocated?
3. Where are the areas of greatest customer density?
4. Should promotional efforts be concentrated in a handful of relatively well-defined areas?
5. Is the market area on weekends the same as the market area for weekdays?
6. Does the furniture department of a department store have the same market area as the men's wear department?

These questions are just a few examples of the many questions which require estimation of the geographic distribution of customers to facilitate accurate analysis.

How can the geographic customer density be estimated? If there are only a handful of customers then this is not an issue. Each customer may be studied individually. More

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common, however, is the situation in which there are a large number of customers (or potential customers), and some convenient way must be devised to summarize the customer distribution.

How the customer distribution is analyzed depends upon whether it is the customers or the potential customers of an organization which are being evaluated. To see this distinction, imagine that a store exists on the North side of a city. That store's customer density probably consists mostly of North side people who live in proximity of the store. The potential customers in the city, however, will be people from all parts of the city. Thus, care must be taken to define appropriately the population of interest before analyzing the customer density. For convenience, we refer to the "distribution of customers" which pertains to the population of interest, even if that population is more accurately "potential customers."

Given that the population is defined appropriately, there are many methods which might be employed to analyze a market. Some methods group customers into discrete geographic subareas ("quadrats") such as zip codes, census tracts, or telephone exchanges. The problem with this approach is that it throws away the information about where the customer is within the subarea, which may sometimes be quite large. The inaccuracies arising from this are analogous conceptually to the inaccuracies which arise from using a histogram to describe a distribution.

An alternative to aggregation of data into subareas is the direct estimation of the customer density. This has the appealing property of not throwing away information. The direct estimation of the customer density may be parametric, assuming a particular distribution (often bivariate normal), or it may be nonparametric, flexibly fitting the data. The latter approach is known in the statistics literature as nonparametric density estimation.

We assume that the researcher has available a random sample of customer addresses, which may be plotted on a map. The addresses may be sampled from internal customer records, or from primary research (e.g., license plate surveys). Given a geographic scatterplot of customers, the distribution of customers may be estimated. Using the estimated customer density, answers to many important market area analysis questions may be facilitated.

This problem is analogous to the determination of density of consumer ideal points in a perceptual map (Donthu and Rust 1988). The main contribution of this paper is to demonstrate the application of nonparametric density estimation to customer location distribution studies. Simulation results show that the error in density estimation using the kernel density estimation method is significantly lower than other popular density estimation methods such as histogram, bivariate normal distribution and Squared Surface Density Analysis.

§2 describes existing methods for estimating customer densities, and §3 proposes using the kernel density estimation approach to estimate customer densities, and §4 illustrates how site selection decisions may incorporate the estimated customer density. Conclusions are summarized in §5.

2. Existing Methods

Estimating customer densities has taken on three main forms in the market area analysis literature: estimating market area boundaries, smoothing aggregated data from subareas, and nonparametric density estimation.

2.1. *Estimating Market Area Boundaries*

Objective methods for estimating market area boundaries have been proposed by Huff and Batsell (1977) and Huff and Rust (1984), with the second method being a refinement of the first. The methodological tool employed in both studies was cubic splines (Ahlberg, Nilson and Walsh 1967, 1978).

Both methods seek to create a smooth boundary which encloses a given percentage (e.g. 80%) of a store's customers. From the standpoint of density estimation, it is as though there were a uniform customer density within the boundary, and presumably a steep drop-off in density at the boundary. The boundary might be considered to be a contour line, in the way that it would be used operationally.

The major problem with this approach is that all information about how customers are distributed is lost, both inside and outside the boundary. Thus, while construction of a market area boundary may be useful in giving a rough idea of a market area's size and shape, it is not particularly useful in guiding decisions such as site selection, which require more precision.

2.2. *Smoothing Aggregated Data*

The inputs to these procedures are the relative frequencies in the subareas. Typically, the height of the density at the centroid of a subarea would be assumed to be proportional to the subarea's relative frequency, and inversely proportional to the subarea's size. Thus the number of data points to which the surface is fit is equal to the number of subareas.

Trend surface analysis (Ripley 1981) has been used in this context to estimate the market area density (MacKay 1973; Peterson 1974). In effect trend surface analysis may be thought of as a multiple regression analysis in which the X and Y coordinates, and their higher order terms, are the independent variables and the observed density is the dependent variable. Trend surface estimates have been refined through the use of wavelength filtering (MacKay 1973), a two-dimensional time series technique.

The major drawback of the aggregation methods is the fact that information about the individual data points is lost. Data may be distributed irregularly within subareas, creating large inaccuracies when the data are aggregated.

2.3. *Nonparametric Density Estimation*

This is the newest approach to estimating customer densities. Rust and Brown (1986) proposed a new method called Squared Surface Density Analysis (SSDA) which estimates the market area density directly from the sampled customer locations. SSDA essentially involves a maximum likelihood estimation of a squared trend surface. It overcomes the trend surface limitations of aggregation bias by using the data points directly. Use of logistic transformations of the original (X, Y) coordinates, in conjunction with the squared form, alleviates logical absurdities such as negative densities or infinitely increasing densities. The method proposed in this paper, kernel density estimation, is also a nonparametric density estimation approach.

3. Kernel Density Estimation

Kernel density estimation is probably the most widely used method of smoothing unidimensional or multidimensional sample data into a continuous probability density function. Estimating a density using kernel density estimation requires, in addition to the data, a "kernel function" K and a smoothing parameter h. The estimated density is generally insensitive to choice of kernel function, but may be strongly affected by the value of the smoothing parameter. If the smoothing parameter is large, then small irregularities are obscured, while a small smoothing parameter results in a bumpy density surface.

Practical applications of kernel density estimation are still very limited, perhaps due to the intimidatingly theoretical nature of the literature in that area. For example, B. W.

Silverman, perhaps the foremost expert on nonparametric density estimation, has said that "A particularly disappointing feature of the technical nature of much of the literature on density estimation is that it may even have had a negative effect, by scaring off potential users of the methods. . . ." (1985). Our view is that application of kernel density estimation to areas of obvious potential use in Marketing is long overdue.

Rosenblatt (1956) first explicitly introduced the kernel estimate, defined in d dimensions as:

$$\hat{f}(\mathbf{x}) = (nh^d)^{-1} \sum_{i=1}^{n} K((\mathbf{x} - \mathbf{x}_i)/h)$$

where K is the kernel function, h is the smoothing factor or window width, n is the number of data points, and x_i are the data points.

The above formulation, while relatively simple, obscures the conceptual simplicity of the method. A physical analogy may be useful in clarifying how kernel density estimation works. Imagine the floor of a room, with dots painted to represent the scatterplot of the sample observations. Now imagine that a large shovelful of sand is dumped on each dot. For an isolated dot, the sand will tend to be deepest at the dot, and slope away as the distance from the dot increases.

On the other hand if dots are close together, then sand will tend to form a large pile between the dots, where the shovelfuls overlap. The depth of the sand at any point in the room is analogous to the estimated density. The shape of an isolated pile of sand left by the shovel is analogous to choice of kernel function, and how flat (smoothed out) each pile is is analogous to how big the smoothing factor h is. Just as the surface of sand in the room will, in general, not resemble any individual shovelful, the density surface estimated by kernel density estimation will, in general, not resemble the kernel function.

Each kernel function $K(\mathbf{x})$ has the following properties:

1. $0 < K(\mathbf{x}) < C$,
2. $K(\mathbf{x}) = K(-\mathbf{x})$,
3. $\int K(\mathbf{x})d\mathbf{x} = 1$,
4. $\int K(\mathbf{x})x\, d\mathbf{x} = 1$,
5. $h \rightarrow 0$ as $n \rightarrow \infty$.

Hence each kernel has a unit area under it, and the smoothing factor approaches zero as the sample size increases. Many distributions, such as normal, uniform, and some special polynomial functions, satisfy the above conditions, and are possible candidates for the kernel function.

Rosenblatt (1956) showed that the class of estimates is pointwise and integratedly consistent, provided that the optimal h is chosen such that as the sample size increases, h tends towards zero. Parzen (1962), imposing further constraints on the kernel function, showed asymptotic unbiasedness. Most of the properties of the kernel function $K(\cdot)$ are transferred to the density function $\hat{f}(\cdot)$. Hence if the kernel function is smooth and is a density, $\hat{f}(\cdot)$ will also be a density and smooth (Fryer 1977).

If the kernel function is defined for all values of $\mathbf{x}$, then $\hat{f}(\mathbf{x})$ has a positive density over the entire space. The density is the sum of n kernel functions, each centered at the observed points. Regions of the space in which there are many data points have a high density, while regions in which there are few points have a low density. The actual estimation, however, is continuous and not by regions.

The choice of an optimal kernel function was considered by Epanechnikov (1969) and Silverman (1978), who showed that the choice of the kernel function is not very crucial, and most functions, including normal and uniform, give near optimal results, even with small sample sizes. They show that the curve estimate $\hat{f}$ approximates the true density $\hat{f}$ even with small sample sizes, and is not dependent on the choice of the kernel function K. Uniform consistency—that is convergence in probability of sup $|\hat{f}(x) - \hat{f}(x)|$ to zero—has been shown to exist.

The smoothing parameter h dictates to what extent the density surface will follow the data. A large h results in a smooth-looking surface, while a small h results in a surface which is more lumpy. The problem of objectively choosing the value of h is a subject of current research, and several objective and partially objective methods have been suggested; see for example, Woodroofe (1970), Parzen (1979), and Silverman (1978 and 1986).

For the purposes of this paper, we choose as kernel function a standard bivariate normal distribution. We choose the value of the smoothing parameter h which would be optimal if the data actually arose from a bivariate normal distribution (Silverman 1986). More specifically we choose the h which would minimize the approximate mean integrated square error of the estimated density, assuming that a bivariate normal distribution generated the data. Of course, we have no reason to actually expect the underlying distribution to be bivariate normal, but this value should provide a reasonable "ballpark" estimate for most distributions which might be encountered in practice.

Kernel density estimation techniques have been used for many years in statistics and mathematics for exploratory analysis, confirmatory analysis, and data presentation. The technique suggests itself naturally for flexibly estimating the density of consumer locations, leading to (for example) methods for optimal site selection and market area comparisons.

A simulation study was designed to establish the superiority of kernel density estimation in its ability to recover multi-modal distributions with minimum amount of error. One hundred data points were generated randomly from either a uniform distribution, a bivariate normal distribution, or mixtures of two or three bivariate normal distributions. Thus the four conditions represented no modes, one mode, two modes, or three modes. The density function values were then estimated over a grid of points, and compared to the actual underlying density values at the grid points.

Table 1 summarizes the design of this simulation study, designed to compare histograms, bivariate normal distribution, SSDA, and kernel density estimation. Two error measures, the sum of the absolute errors and the maximum absolute error, were calculated for each of ten replications. The means for each error measure, for each number of modes, for each estimation method, are shown in Table 2.

The results clearly show that kernel density method is superior to all methods compared here. The kernel method has the minimum mean sum of absolute errors and the minimum mean maximum absolute error across all cases simulated here except for the mean max-

TABLE 1

Design of Simulation Study to Compare Density Estimation Methods

	Competing Methods
Histogram	2, 4, 8, 16, 32 windows/dimension
Bivariate Normal	
SSDA	
Kernel	
	Modes
Zero	Uniform distribution
One	Mode at (0.5, 0.5)
Two	Modes at (0.25, 0.75) and (0.75, 0.25)
Three	Modes at (0.25, 0.75), (0.75, 0.25), and (0.25, 0.25)

TABLE 2

Comparisons of Density Estimation Techniques

Method / Data	Histogram (# of windows per dimension) 2	4	8	16	32	Bivariate Normal	SSDA	Kernel
Mean Sum of Absolute Errors								
Uniform dist	1250.0	1874.6	2185.5	2343.6	2420.3	1132.8	1917.0	226.3
One mode	2981.2	1995.5	2119.6	2258.5	2324.9	377.1	2083.9	237.7
Two modes	1885.2	1988.8	2094.2	2236.7	2310.2	2836.6	3144.7	233.0
Three modes	1853.5	1973.9	2091.8	2236.1	2309.1	2248.0	2891.5	234.8
Overall (total)	1992.5	1957.7	2122.8	2268.7	2341.3	1648.6	2557.4	232.9
Mean Maximum Absolute Error								
Uniform dist	0.60	0.90	1.00	1.00	1.00	0.97	2.30	0.36
One mode	15.38	15.06	14.93	15.29	15.59	2.43	15.74	3.35
Two modes	7.04	7.57	7.55	7.82	7.30	8.80	7.91	0.86
Three modes	4.73	5.11	5.10	5.23	5.29	4.27	5.28	0.75
Overall (total)	6.94	7.16	7.15	7.33	7.45	4.19	8.17	1.33

imum absolute error in the uni-mode situation. Except for this one case the errors by kernel density method is significantly lower than all other methods (at the 0.05 level). In the cases where the points are distributed bi-modally or tri-modally the kernel method of density estimation dominates over all other methods.

As expected, the bivariate normal method of density estimation performs well in capturing the uni-modal data. The SSDA method does not perform as well as the kernel method, but for most situations the error is comparable to that of histograms and the bivariate normal distribution.

In addition, Donthu and Rust (1988) have shown that using the kernel density estimate generally works better (based on simulations) than using the sample data points directly, when estimating market share. Thus the current practice of using the sample data points directly, skipping the density estimation step, may be ill-advised.

Computer programs for implementing kernel density estimation are very simple to write. In addition, many such programs are readily available, some of which speed up computations by using fast Fourier transforms. A listing for a kernel density estimation program, which includes the option of preparing market area data for plotting, is available from the authors (Donthu and Rust 1988b).

4. Site Selection

The choice of a store's location is perhaps the single most important decision a retailer has to make. The site selection decision is, in fact, not restricted to retailing, but is also crucial for other location decisions such as industrial location, location of a hospital, and locating of services such as banks, schools, etc.

All of these decisions involve the optimal selection of a site given the locations of consumers and competition. Many approaches have been developed to address this problem. For example, Central Place Theory (Christaller 1935), the microeconomic approach (Losch 1954), Multiplicative Competitive Interaction Models (Nakanishi and Cooper 1974; Jain and Mahajan 1979), the direct utility assessment approach (Louviere and Henley 1977), the checklist method (Applebaum 1965), and the analog approach (Cohen and Applebaum 1960) are just a few of the methods proposed for establishing the best site for a distribution center. Craig, Ghosh, and McLafferty (1984) provide an excellent review of existing methods.

Most site location models rely upon the concept of spatial demand. Gravitation models (Reilly 1931) and their extensions (Huff 1964) employed the hypothesis that demand was inversely related to distance to be traveled. Recent work has refined the relationship between distance and other explanatory variables on spatial demand (e.g. Black, Ostlund and Westbrook 1985; Ingene 1984; Louviere 1984; Bucklin 1967; Fotheringham 1980), but, the most important single variable has typically been distance (Arnold, Oum and Tigert 1983).

Site selection models thus have two key components. They must determine where the potential customers are located, and then optimize the location of the distribution center to maximize patronage. Spatial demand models are well-established for estimating market share. Density estimation complements spatial demand models by estimating the distribution of customer locations, and thus supplying the most important variable, distance.

The remainder of the section considers a simple illustration of how kernel density estimation may be used in the site selection decision. The example shown deliberately avoids many of the variables which ordinarily complicate the site selection decision, in order to highlight the application of nonparametric density estimation. This general density approach may also be used in more complicated situations, such as the multiple store location problem (Achabal, Gorr and Mahajan 1982; Ghosh and Craig 1983). Using kernel density estimation for site selection is conceptually analogous to using kernel methods for product positioning (Donthu and Rust 1988) and several of the equations are essentially equivalent.

4.1. *Method*

It is assumed that an emergency health care supplier is trying to ascertain the best location for a new hospital emergency facility. Because only emergency care will be provided it will be assumed that no elective care will take place at the facility. Thus it is not necessary to employ a large number of variables in a choice model, as is typically done in the general hospital selection model (Folland 1983).

The population of interest is all potential hospital patients in the metropolitan area, rather than the patients of any individual hospital. Thus the customer distribution should be estimated from a sample derived from the population at large.

We also assume, for simplicity of exposition, that the probability of patient i ending up at facility j is proportional to an exponential function of the distance between facility j and the residence of patient i, which results in a Luce formulation of the probability of patient i arriving at facility j (Luce 1959; Schonemann and Wang 1972):

$$P_j(x_i, y_i) = U_{ij} / \sum_{k=1}^{K} U_{ik}$$

where K is the total number of facilities in the market, and

$$U_{ij} = \exp(-Cd_{ij}^2)$$

where d_{ij} is the distance between patient i's residence and facility j, C is a constant, and (x_i, y_i) is the location of patient i's residence.

The constant C may be calibrated from the market shares of the existing facilities. One simple method would be to conduct a numerical search for the value of C which minimized the sum of squared differences between the estimated and actual market shares. Because the search over C is in only one dimension, computational requirements are not excessive. In this example the market shares were not readily available, so for purposes of illustration we assumed a value of $C = 1$. In an actual application market share data (or estimates) would be used to obtain the optimal value of C.

This formulation is analogous to that of a consumer choice model. In that context U_{ij} represents the utility of object j to subject i, whereas in this context U_{ij} instead refers to a relative likelihood of patient i arriving at facility j.

Batsell and Lodish (1981), Currim (1982), Fotheringham (1985), and Kamakura and Srivastava (1984) have discussed the theoretical limitations of this formulation and have proposed several extensions and revisions. We believe it should be possible to derive similar results from alternative choice specifications and that this is not crucial to our analysis.

The market share of facility j may then be calculated directly if we know the patient density $g(x, y)$. We assume, for the sake of simplicity, that each resident of the city has an equal probability of becoming a patient. This assumption may easily be relaxed if additional information is available concerning the relative incidence of emergency health

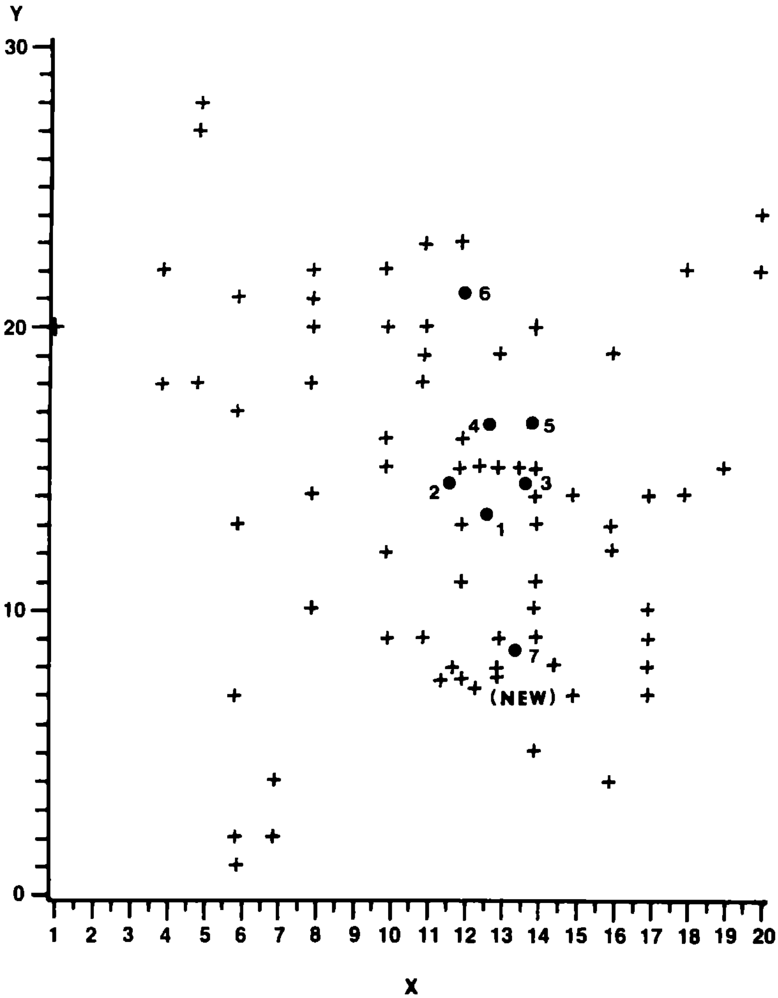

FIGURE 1. Scatter Plot of Consumers (100) and Hospitals (6).

problems in various geographic areas. For example, a random sample of ambulance destinations from the city's hospitals might instead be used to estimate the patient density.

Given patient density $g(x, y)$, the market share MS_j for facility j is estimated to be:

$$MS_j = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) P_j(x, y)\, dx\, dy.$$

This expression takes the existing locations into account, because the expression for $P_j(x, y)$ is a function of all competing locations, in addition to the location for facility j.

It has been shown (Donthu and Rust 1988) that the coordinates which maximize market share for facility j may be approximated using a Taylor series expansion of the above double integral. The approximately optimal location (x_j^*, y_j^*) is the solution of the simultaneous equations:

$$\gamma x_j^* + \beta y_j^* + \alpha_x = 0,$$

$$\gamma' x_j^* + \beta' y_j^* + \alpha_y' = 0, \qquad \text{where}$$

$$\alpha = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y)(dP/dm)\, dx\, dy,$$

$$\alpha' = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y)(dP/dn)\, dx\, dy,$$

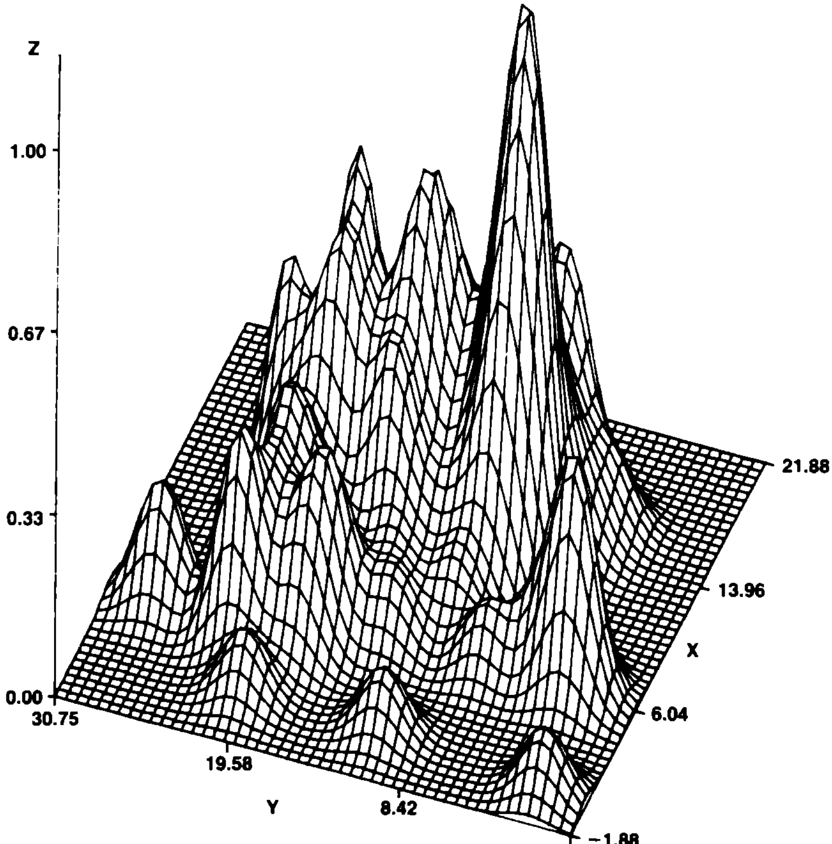

FIGURE 2. Kernel Density Plot of Consumers.

$$\beta = \beta' = (1/2) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y)(d^2P/dn^2)\,dx\,dy,$$

$$\gamma = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y)(d^2P/dm^2)\,dx\,dy,$$

$$\gamma' = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y)(d^2P/dn^2)\,dx\,dy$$

and (m, n) is a starting estimate of the optimal location.

This procedure may be performed iteratively, which becomes a Newton search procedure. On each step the (m, n) starting point is replaced by the (x_j^*, y_j^*) from the previous step. As usual with such procedures, multiple starting points may be used, to reduce the probability of finding only local optima.

4.1. *Example*

The method is illustrated on data from a large Southeastern city which had six major existing emergency facilities, all in hospitals. A random sample of 100 addresses was

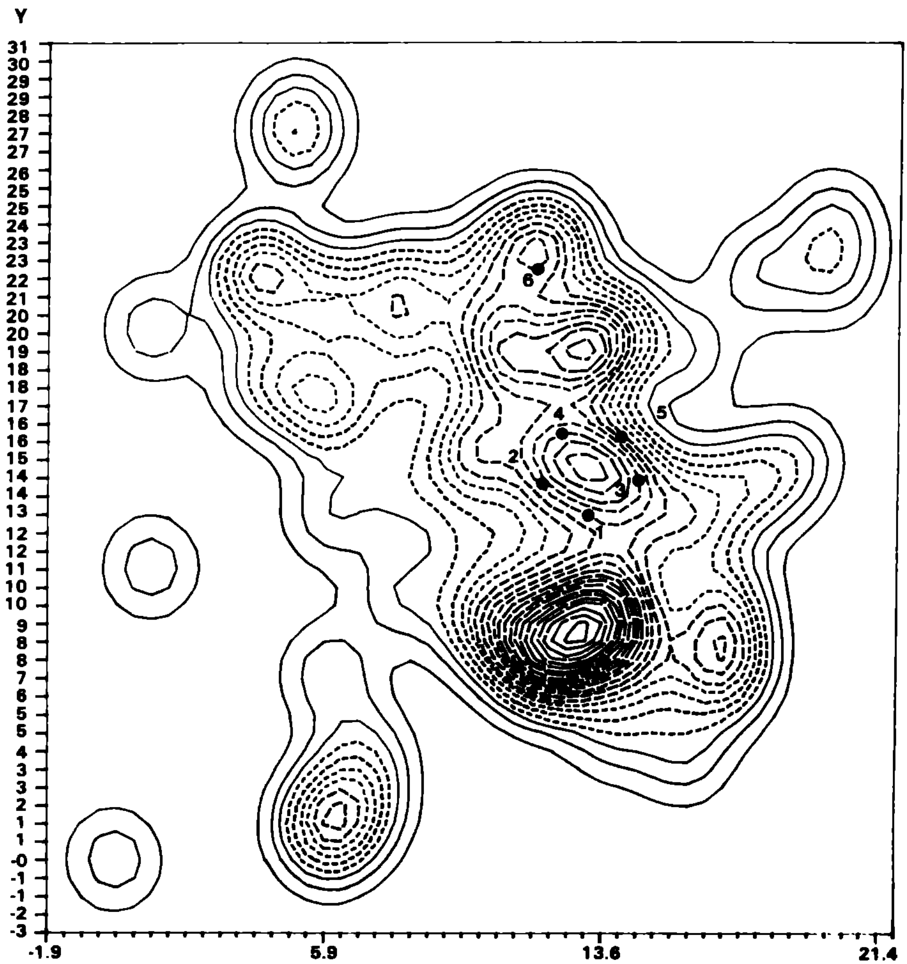

FIGURE 3. Contour Plot of Consumer Density.

drawn from the phone book (other sample frames might just as easily have been used), for the purpose of estimating the city's population geographic density surface. Figure 1 shows the scatter-plot of residence locations, obtained from superimposing a coordinate grid on a street map.

Figure 2 shows a three-dimensional surface which depicts the city's geographic population density. Figure 2 is obtained by calculating the density (Z axis) at each point of a fine (X, Y) grid, and then submitting the (X, Y, Z) plot points to a three-dimensional plot routine (SAS GRAPH, in this case). Figure 3 shows the corresponding contour plot. It is apparent by inspection that five of the six existing hospitals are clustered together in what the street map reveals to be the downtown area, with the sixth hospital off to the North. From Table 3 we can see that facility six's advantageous positioning, away from competition, should represent a competitive advantage.

Figure 1 shows the preferred location of the new emergency facility, facility seven. By locating to the South, the new facility avoids competition and should obtain a large market share. Table 3 shows that the central hospitals should suffer the greatest percentage decrease in market share. The market share predictions could be improved by adding variables such as size of hospital, cost, experience of doctors, etc. However, for illustrative purpose, we have included only one variable, distance, as we do not attempt to compare the market share predictions by different methods.

5. Conclusions

Kernel density estimation provides an attractive method for estimating flexibly the geographic distribution of customers. Whatever the shape of the underlying density surface, kernel is capable of fitting that shape, given a large enough sample size.

Unlike most other approaches to estimating market area densities, kernel density estimation does not throw away location information through aggregation into subareas (quadrats). Being a nonparametric approach, it does not assume any particular functional form. It also outperforms the SSDA method, the nonparametric density estimation method which has previously been applied to market area analysis.

It is shown how a density-based product positioning methodology can be applied to the analogous issue of site selection. This provides the conceptual basis for more sophisticated models of site selection and spatial demand which could use a more complete set of predictor variables.

The methods illustrated in this paper indicate how kernel density estimation may be used to estimate market area densities, and provide an objective basis for decision-making.

TABLE 3

Hospital Market Share Estimation

Hospital No.	Estimated Market Share Before #7 Entered the Market	Predicted Eventual Market Share After #7 Entered the Market
1	15.6	10.5
2	15.4	10.6
3	15.2	11.0
4	14.3	10.8
5	14.0	10.7
6	25.5	22.2
7	—	24.2

Kernel density estimation also provides a powerful visual approach to analyzing a market area.[1]

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[1] This paper was received in April 1987 and has been with the authors 3 months for 2 revisions.

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