Mean-Variance Spanning Tests
With Short-Sales Constraints

Farid AitSahlia[1] Thomas Doellman[2] Sabuhi Sardarli[3]

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Abstract

We address implementation issues related to Wald tests associated with mean-variance spanning when short positions in portfolios are prohibited. In particular, we exploit the uniqueness of the stochastic discount factor in the presence of a risk-free rate to avoid potential numerical stability issues. We also show that tests that have appeared in the literature on retirement plans in the U.S. are inaccurate and that their proper implementation leads to significantly different results.

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[1] Warrington College of Business, University of Florida, Gainesville, FL, 32611; email: farid1@ufl.edu ; Tel: +1 (352) 392-5058.

[2] John Cook School of Business, Saint Louis University, St. Louis, MO 63108; email: thomas.doellman@slu.edu, Tel: +1 (314) 977-3815.

[3] College of Business Administration, Kansas State University, Manhattan, KS 66506; email: ssardarli@ksu.edu ; Tel: +1 (785) 532-5771.

1. Introduction

A set of assets is said to span the mean-variance space if the efficient frontier it generates cannot be improved upon with additional assets (Huberman and Kandel, 1987). Mean-variance spanning has been used to, among others, assess the benefits of international diversification, evaluate mutual fund performance, test linear-factor asset pricing models and consider the relevance of cryptocurrencies as an alternative asset class (cf., Errunza et al. (1999); Li et al. (2003); DeRoon et al. (2001); Fama and French (2015, 2018); Petukhina et al. (2021).)

Empirically testing for mean-variance spanning is conducted via multivariate regression, going back to the seminal paper of Huberman and Kandel (1987). It is extensively studied in Kan and Zhou (2012), who incidentally point out a misprint for a crucial F-statistic in Huberman and Kandel (1987) that has been indiscriminately reprised in a number of subsequent papers.

The mean-variance literature has relied mostly on the absence of short-sales and transaction costs until DeRoon et al. (2001) (DNW, hereafter) where it was first addressed. Their approach was applied by Elton et al. (2006) and Tang et al. (2010) (henceforth, EGB and TMMU, respectively) to evaluate whether defined contributions retirement plans, such as 401(k) plans in the United States, efficiently offer investment options that cater to a wide spectrum of risk profiles. In their implementations, both EGB and TMMU use excess returns in the regression specification, in contrast to the raw returns used in most of the mean-variance spanning literature. In this paper we show that their testing hypothesis is incorrect and that it leads to overstated results.

The remainder of the paper is organized as follows. In Section 2, we review the methodology of mean-variance spanning with short-sales constraints and address implementation issues associated with the related Wald tests, particularly their potential numerical instability. In Section 3, we show how these tests are incorrectly implemented in the extant literature regarding the efficiency of defined contributions retirement plans in the United States. Section 4 provides an empirical illustration of the resulting discrepancy relative to a proper implementation. Section 5 concludes.

2. A Re-Examination of Mean-Variance Spanning Tests Under Short-Sales Constraints

We review the regression-based mean-variance spanning test methodology when short sales are prohibited and highlight some issues related to its implementation as presented in DNW. Due to the multitude of stochastic discount factors, DNW suggest using only the smallest and the largest mean discount factors. However, these are not observable and must be implied. As an alternative to their approach, we instead appeal to the uniqueness of the mean discount factor in the presence of a risk-free rate, with the former being the inverse of 1 plus the latter.

We start by making clear basic return terminology. Between dates t and $t+1$, return refers to simple net (raw) return, which is defined as $(P_{t+1}+I_t)/P_t-1$; where P_t, P_{t+1}, I_t are, respectively, the asset value at times t and $t+1$, and the related income in that interval.

To assess the efficiency of *K* assets relative to a benchmark of *N* other assets (e.g., index funds), we want to determine whether the mean-variance efficient frontier associated with the *K* assets coincides with that generated with the augmented set of *K*+*N* assets. In other words, the *K* assets are "sufficient" to span the efficient frontier of the *K*+*N* assets.

Let R and r be the $K\times 1$ and $N\times 1$ return vectors, respectively, of the *K* assets and the *N* benchmark indices. Denote by μ_R and μ_r their corresponding expected return vectors. Related covariance matrices are defined as follows:

$\Sigma_{R,R}$: $K\times K$ covariance matrix between the K assets with returns vector *R*;

$\Sigma_{r,r}$: $N\times N$ covariance matrix between the returns of the *N* benchmark indices captured by vector *r*;

$\Sigma_{R,r}$: $K\times N$ covariance matrix of the *K* asset returns with the *N* benchmark returns.

$\Sigma_{r,R}$: $N \times K$ transpose[1] of $\Sigma_{R,r}$ (i.e., $\Sigma_{r,R} = \Sigma'_{R,r}$)

These covariance matrices are then concatenated across all $K + N$ assets into the $(K + N) \times (K + N)$ covariance matrix Σ defined as:

$$\Sigma = \begin{pmatrix} \Sigma_{R,R} & \Sigma_{R,r} \\ \Sigma_{r,R} & \Sigma_{r,r} \end{pmatrix}.$$

Similarly, we denote by $\mu \equiv \begin{pmatrix} \mu_R \\ \mu_r \end{pmatrix}$ the $(K + N) \times 1$ (concatenated) vector of expected returns across the $K + N$ assets.

Given short-sales constraints, the mean-variance optimization problem across the *K*+*N* assets consists in determining the K-dimensional vector $\omega_R \geq \mathbf{0}_R$[2] and the *N*-dimensional vector $\omega_r \geq \mathbf{0}_r$ that maximize

$$(\omega'_R, \omega'_r)\begin{pmatrix} \mu_R \\ \mu_r \end{pmatrix} - \frac{1}{2}\gamma(\omega'_R, \omega'_r)\begin{pmatrix} \Sigma_{RR}\Sigma_{Rr} \\ \Sigma_{rR}\Sigma_{rr} \end{pmatrix}\begin{pmatrix} \omega_R \\ \omega_r \end{pmatrix}, \quad (1)$$

subject to $\omega'_R \cdot i_R + \omega'_r \cdot i_r = 1$. If the assets in the plan span, then the optimal mean-variance allocation (ω^*_R, ω^*_r) is such that $\omega^*_r = 0_{\mathrm{N}}$.

Starting with Huberman and Kandel (1987), the formal empirical analysis of (1) above has been tied to the multivariate regression specification

$$r = \alpha + \beta R + \varepsilon \quad (2)$$

Accounting for short-sales restrictions, DNW derive the necessary and sufficient conditions for mean-variance spanning (cf. (15) on their page 727):

$$\upsilon\, \alpha + \beta i_R - i_r \leq 0, \quad (3)$$

[1] We use the superscript $'$ for matrix transposition.
[2] $\mathbf{0}_R$ is a vector of zeros with same dimension(K) as the vector R. In the remainder of the paper, we do not use boldface and we omit the subscript to reference dimension if the context is evident.

where υ is the mean of any stochastic discount factor that prices the assets. Given the range of values for υ, DNW suggest (pp. 729-730) that, for spanning, it is enough to jointly test

$$\begin{aligned} &1\,\alpha + \beta i_K - i_N \leq 0 \\ &\upsilon_{min}\,\alpha + \beta i_K - i_N \leq 0, \end{aligned} \tag{4}$$

where 1 is the upper bound on the values of υ and $\upsilon_{min} = \frac{1}{E[R^{GMV}]}$ is their lower bound, where $E[R^{GMV}]$ is the mean gross return (i.e., 1 + net return) of the global minimum variance portfolio.

To test the inequalities in (4), DNW follow Kodde and Palm (1986) and use the Wald statistic

$$\xi = \min_{\gamma \geq 0} (\tilde{\gamma} - \gamma)' \tilde{\Sigma}^{-1} (\tilde{\gamma} - \gamma), \tag{5}$$

where

$$\tilde{\gamma} = \begin{pmatrix} -\hat{\alpha} - \hat{\beta} \times i_K + i_N \\ -\frac{1}{1+\mu}\hat{\alpha} - \hat{\beta} \times i_K + i_N \end{pmatrix}, \tag{6}$$

with $\mu = E[R^{GMV}] - 1$, and

$$\tilde{\Sigma} = \begin{pmatrix} -I_N & -A \\ -\dfrac{1}{1+\mu} I_N & -A \end{pmatrix} \Omega \begin{pmatrix} -I_N & -A \\ -\dfrac{1}{1+\mu} I_N & -A \end{pmatrix}', \tag{7}$$

with I_N defined as the $N \times N$ identity matrix, A as the Kronecker product $I_N \otimes i_K{}'$, and Ω as the $(N + NK) \times (N + NK)$ covariance matrix between the multivariate intercept α and the loading matrix β in the regression (2), and where $\hat{\alpha}$ and $\hat{\beta}$ refer to estimates of α and β, respectively. Note that estimates for μ are typically two orders of magnitude smaller than 1. Therefore, when they are indeed very small for global minimum variance portfolios, as often occurs, the first N rows (resp. columns) in the matrix pre- (resp. post) multiplying Ω are almost identical to the latter N, making $\tilde{\Sigma}$ nearly singular and resulting frequently in non-computable inverses, as we experienced in our empirical implementation on 401(k) plans. As a result, we instead appeal to the fact that in the presence of a risk-free rate, say r_f, there is only one stochastic discount factor, with mean $\frac{1}{1+r_f}$. Consequently, instead of the two sets of inequalities in (4), we only need to deal with one in (3), where $\nu = \frac{1}{1+r_f}$, and for (6) and (7), we now have:

$$\tilde{\gamma} = -\frac{1}{1+r_f}\widehat{\alpha} - \hat{\beta} \times i_K + i_N$$

$$\tilde{\Sigma} = \left(\frac{1}{1+r_f} I_N \;\; - A\right)\Omega\left(\frac{1}{1+r_f} I_N \;\; - A\right)'$$

3. A Critique of the Current Literature on 401(k) Plan Spanning Tests

In the presence of short-sales constraints, both EGB and TMMU use excess returns relative to the risk-free rate to test the null

$$\alpha^* \leq \mathbf{0}, \tag{8}$$

where the N-dimensional α^* is a Jensen-type alpha, that is $\alpha^* = (\mu_r - r_f) - \beta(\mu_R - r_f)$. While not using any formal optimization model, EGB justify their choice by arguing that "*If short sales are forbidden, then only the addition of an asset with positive alpha can improve the efficient frontier*…" (p. 1304). Similarly, TMMU state: "*As short-sales are not allowed for market benchmark index, if none of the* α_i *are statistically significantly positive, we could conclude that performance of funds under the plan cannot be improved by holding a long position in any of the eight market benchmark indices*." (p. 1078).

We argue next that (8) is in fact not reflective of an optimality condition with short-sales constraints, in the sense that one can still improve the efficient frontier of a given set with the addition of another asset whose alpha is negative relative to the former. Furthermore, we show that it does not correspond to the spanning condition (3) above in raw returns.

First, recall that the covariance matrices of excess returns and raw returns are the same, and that we retrieve the same variance-minimizing portfolios subject to expected returns constraints whether the optimization problem is stated in raw returns or excess returns[1]. Consider now assets 1, 2, and 3 with return covariance matrix

$$\begin{pmatrix} 0.011 & 0.002 & 0.001 \\ 0.002 & 0.012 & 0.003 \\ 0.001 & 0.003 & 0.020 \end{pmatrix}$$

and their associated expected excess returns $\mu_1 - r_f = 0.043$, $\mu_2 - r_f = 0.001$, and $\mu_3 - r_f = 0.028$, respectively. We then have $\mu_2 - r_f = -0.012 + 0.1689 \times (\mu_1 - r_f) + 0.1416 \times (\mu_3 - r_f)$. For a given expected excess return of 3%, the variance minimizing allocation when only assets 1 and 3 are considered is 0.1333 and 0.8667, with a resulting standard deviation of portfolio return equal to 0.1243. On the other hand, with the addition of asset 2, and for the same level of expected excess return of 3%, the variance minimizing strategy is 0.5410, 0.2265, and 0.2326, for assets 1, 2, and 3, respectively. The standard deviation of the return on this portfolio is smaller, namely, 0.0773, despite the negative alpha of the additional asset 2.

We now show that, for excess returns, (8) does not correspond to condition (3), when re-expressed in terms of raw returns. Based on raw returns, regression (2) yields

$$\beta = \Sigma_{\mathrm{r,R}} \Sigma_{R,R}^{-1} \tag{9}$$

Letting $r^* = r - r_f i_r$ and $R^* = R - r_f i_R$ denote the excess returns associated, respectively, with r and R, the corresponding regression

$$r^* = \alpha^* + \beta^* R^* + \epsilon^* \tag{10}$$

leads to

$$\beta^* = \Sigma_{r^*,R^*} \Sigma_{R^*,R^*}^{-1}, \tag{11}$$

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[1] since $\omega'(\mu - r_f) = \bar{\mu} - r_f$ is equivalent to $\omega'\mu = \bar{\mu}$ when $\omega'\mathbf{1} = 1$, where μ is a vector of expected raw returns, $\bar{\mu}$ a target expected raw return, r_f a risk-free rate of return, and $\mathbf{1}$ a vector of elements all equal to 1.

where the matrices are the excess returns equivalent of those in (9) above. Clearly, since r_f is deterministic, $\beta = \beta^*$. From (2) and (10) we can infer

$$\alpha^* = \alpha + r_f(\beta i_R - i_r).$$

With $\nu = \frac{1}{1+r_f} > 0$, condition (8) is equivalent to

$$\nu\alpha + \frac{r_f}{1+r_f}(\beta i_R - i_r) \leq 0, \tag{12}$$

which is not exactly (3), even though given that r_f being significantly smaller than 1, the two conditions could approximately be similar. We show next that, in fact, the discrepancy in the test results can be significant in practice.

4. Empirical Illustration

Our sample consists of 7,975 DC plans with data provided by Brightscope, Inc., an information provider of retirement plan ratings and investments analytics, with returns covering the years 2004-2008, which overlaps with the periods covered by both EGB and TMMU. This sample is significantly larger than the 417 plans of EGB and the 1,003 plans of TMMU. Return data for benchmark funds are from DataStream. Our benchmark set is identical to the ones used by EGB and TMMU, consisting of Barclays Capital Aggregate Bond Index, Credit Suisse High Yield Bond Fund, and Citigroup World Government Bond Non-US$ Index for returns of fixed income securities; the Russell 1000 Growth, Russell 1000 Value, Russell 2000 Growth, and Russell 2000 Value for returns of large-, mid- and small-cap equities; and the MSCI EAFE Index for international exposure.

For a given plan, a rejection of the null hypothesis suggests, at the 5% level of confidence, that its mean-variance spanning frontier can be improved with additional funds from the benchmark set. When the null is not rejected, for the sake of expository simplification, especially when comparing our results with those of EGB and TMMU we keep the label of ‘spanning’ that they adopted, instead of the strictly correct “spanning not rejected.” We find that 46% of plans span, much lower than the 53% and the 97% rates found in Elton et al. (2006) and

Tang et al. (2010), respectively. While this difference may be explained by our vastly larger and more diverse data set, it is also due to the fact that both sets of authors test the null $a \leq \mathbf{0}$, which is only a necessary --but not sufficient -- condition. As a result, when they fail to reject the null, they may incorrectly conclude in favor of spanning.

5. Conclusion

Regression-based mean-variance spanning tests are ubiquitous in empirical finance. Our paper centers on challenges that arise when these tests are implemented in the context of short-sales constraints, as in defined-contribution retirement plans such as the 401(k) plans for U.S. employees. While the standard Wald testing methodology relies on the implied means of unobservable discount factors, we exploit the property that the mean discount factor is uniquely determined in the presence of a risk-free asset to help with its efficient implementation, thus avoiding potential issues of numerical instability. We also show how their incorrect implementation in the retirement literature has led to vastly overstated results.

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