Pricing and Hedging of American Knock-In Options

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American barrier options of the knock-in type involve non-Markovian optimal stopping problems for early exercise. They therefore cannot be priced via standard methods such as binomial or trinomial trees and finite-difference schemes for free-boundary partial differential equations.

This article provides a modified tree method to price these options. It also develops fast and accurate analytic approximations for the price and hedge parameters.

Complex derivatives have become accepted instruments to tailor risk coverage for risk managers and investors. Barrier-type options have become important instruments, particularly for the valuation of structured products (see Banks [1994]). They are also widely used in currency markets.

The holder of a barrier option acquires option coverage on only a subset of the risky outcomes for which a plain vanilla option pays off; this reduces the cost of the resulting coverage so that the holder of the contract does not have to pay for contingencies the holder thinks are unlikely to occur. Because of this flexibility, barrier options were traded over the counter long before the opening of the Chicago Board Options Exchange, and have become some of the most commonly traded derivative contracts. American barrier options offer the added flexibility of early exercise but have to be priced using numerical algorithms, as they do not have closed-form solutions, unlike their European-style counterparts (see Merton [1973] and Rubinstein and Reiner [1991]).

Naive application of the Cox-Ross-Rubinstein binomial tree method for barrier options has been shown by Boyle and Lau [1994] to yield inaccurate values, even with many steps. To address this problem, which stems from the position of the barrier relative to the grid, a number of variants of the tree method have been advanced.

Ritchken [1995] implements a trinomial tree method. Cheuk and Vorst [1996] develop a time-dependent shift for the trinomial tree, and Figlewski and Gao [1999] introduce an adaptive mesh model that grafts high-resolution lattices around points that cause the inaccuracies in the binomial model. In an alternative to tree methods, Gao, Huang, and Subrahmanyam [2000] and AitSahlia, Imhof, and Lai [2003] extend the price decomposition approach originally developed for standard American options by Kim [1990], Jacka [1991], and Carr, Jarrow, and Myneni [1992] to derive analytic approximations for American knock-out prices and hedge parameters.

The sum of American knock-in and knock-out prices does not equal the standard American option price, as is the case for European barrier options. Moreover, the knock-in option value process is non-Markovian, so the classic binomial or trinomial tree methods or numerical partial differential equations for standard American options cannot be applied directly to price American knock-in options. We first explain why this is the case, and contrast American knock-in and knock-out options.

Then we develop a modified binomial tree method to price and hedge American down-and-in puts. An alternative method is based on the price decomposition approach for standard American options. We propose an efficient approximation to implement this approach that leads to analytic approximations that have penny accuracy. Numerical results for both approaches are provided, and we also address the evaluation of hedge parameters.

I. AMERICAN KNOCK-IN AND KNOCK-OUT OPTIONS

Consider an underlying asset whose price process S_t follows a geometric Brownian motion with volatility σ and that pays dividends at rate q in a market environment where the riskless rate of return is r. Let H be a barrier for either a knock-in or a knock-out option. Let T be the expiration date for any option on the asset. Let $g(S, K)$ denote the option's payoff when exercised at asset price S. Then, $g(S, K) = (S - K)^+$ is the payoff of a call option with exercise price K, and $g(S, K) = (K - S)^+$ is the payoff on the corresponding put. Define:

$$\tau_H^{(D)} = \inf\{t \leq T : S_t \leq H\}$$
$$\tau_H^{(U)} = \inf\{t \leq T : S_t \geq H\}$$

That is, $\tau_H^{(D)}$ (or $\tau_H^{(U)}$) is the first time the price of the underlying asset falls below (or rises above) the barrier H. Then, for any stopping (early exercise) time $\tau \leq T$:

$$E_S\left[e^{-r\tau}g(S_\tau, K)\right] = E_S\left[e^{-r\tau}g(S_\tau, K)1_{\{\tau\geq\tau_H^{(.)}\}}\right] + E_S\left[e^{-r\tau}g(S_\tau, K)1_{\{\tau<\tau_H^{(.)}\}}\right] \quad (1)$$

where $E_S[X]$ denotes the expectation of a random variable X conditional on the initial value $S_0 = S$. When $\tau = T$, Equation (1) expresses the well-known relation between the price of a standard European option, on the left, and the prices of corresponding European knock-in and knock-out options, respectively, on the right.

Let $T_{a,b}$ denote the class of stopping times taking values between a and b with $a < b$. From Karatzas [1988], the price of a standard American option is:

$$V(S) = \sup_{\tau\in\mathcal{T}_{0,T}} E_S\left[e^{-r\tau}g(S_\tau, K)\right] \quad (2)$$

and the prices of the corresponding American knock-in and knock-out options are, respectively:

$$V_{\text{IN}}(S) = \sup_{\tau\in\mathcal{T}_{0,T}} E_S\left[e^{-r\tau}g(S_\tau, K)1_{\{\tau\geq\tau_H^{(.)}\}}\right] \quad (3)$$

$$V_{\text{OUT}}(S) = \sup_{\tau\in\mathcal{T}_{0,T}} E_S\left[e^{-r\tau}g(S_\tau, K)1_{\{\tau<\tau_H^{(.)}\}}\right] \quad (4)$$

Since the suprema in Equations (2)-(4) are attained at different stopping times, the price of a standard American option cannot be decomposed as in (1) into the sum of the corresponding American knock-in and knock-out options.

Since $\{S_t, t \geq 0\}$and $\{S_{min(t,\tau_H^{(.)})}, t \geq 0\}$ are Markov processes, the standard American option expressed in Equation (2) and the American knock-out option in Equation (4) are associated with Markovian optimal stopping problems. We can express Equation (4) as $V_{OUT}(S, 0)$, where:

$$V_{\text{OUT}}(S,t) = \sup_{\tau\in\mathcal{T}_{t,T}} E_S\left[e^{-r(\tau-t)}g(S_\tau, K)1_{\{\tau<\tau_H^{(.)}\}}|S_t = S\right] \quad (5)$$

The optimal stopping problem in Equation (3) associated with an American knock-in option, which becomes effective only after $\tau_H^{(.)}$, is non-Markovian. Because of this, we cannot apply standard algorithms such as finite-difference methods for free-boundary PDEs or binomial trees to compute $V_{IN}(S)$. Instead we use the representation:

$$V_{\text{IN}}(S) = \int_0^T e^{-rt}V(H,t)P\left\{\tau_H^{(.)} \in dt|S_0 = S\right\} \quad (6)$$

where $V(H, t)$ denotes the price of a standard American option (with maturity T and strike K) at time t when $S_t = H$.

II. MODIFIED BINOMIAL METHOD FOR AMERICAN KNOCK-IN PUTS

We can compute Equation (6) using a modified binomial tree method. Even faster analytic approximations decompose the price of a standard American option into the sum of the corresponding European option price and an early exercise premium.

To fix the ideas, we consider from now on the case of down-and-in puts. Typically investors use put options as insurance against possible drops in the value of an asset they are holding. A down-and-in put enables its holder to reduce the cost of such insurance by requiring that it be effective only after the asset price falls below a barrier H.

We modify the binomial tree method to compute the price in Equation (6) of an American down-and-in put with $S > H$. While the usual binomial tree starts from the root node S and may not include H as a node value, our modified binomial tree uses a lattice that includes the barrier.

The integral in (6) involves the distribution of the first time that the geometric Brownian motion S_t crosses the level H. Because $S_t = S \exp\{(r - q - \sigma^2/2)t + \sigma B_t\}$ under the risk-neutral measure P, where $\{B_t\}$ is a standard Brownian motion, and because the first-passage density of Brownian motion has a simple formula, we use the change of variables:

$$z = \log S, \quad \gamma = \log H, \quad \lambda = r - q - \sigma^2/2 \tag{7}$$

The knock-in time $\tau_H^{(D)} = \inf\{t \geq 0: S_t \leq H\}$ can then be expressed as $\tau_H^{(D)} = \inf\{t \geq 0: Z_t \leq \gamma\}$, where:

$$Z_t = z + \lambda t + \sigma B_t \tag{8}$$

is a Brownian motion with drift λ. Hence the probability distribution of $\tau_H^{(D)}$ has a density function f_z given explicitly by:

$$f_z(t) = \left(\frac{z-\gamma}{\sigma t^{3/2}}\right) n\left(\frac{\gamma - z - \lambda t}{\sigma\sqrt{t}}\right) \quad \text{for} \quad t > 0 \tag{9}$$

where $n(.)$ is the standard normal density function; see Karatzas and Shreve [1988, p. 196].

Using Equation (9), we can express the value (6) of an American down-and-in put [which we will denote $P_D(S)$ instead of $V_{\text{IN}}(S)$] as:

$$\begin{aligned} P_D(S) &= \int_0^T e^{-rt} V(H,t) f_z(t) dt \\ &\approx \delta \sum_{k=1}^{M} e^{-rt_k} V\left(e^{\gamma}, t_k\right) f_z(t_k) \end{aligned} \tag{10}$$

Here the approximating Riemann sum involves $M + 1$ equally spaced time steps $t_0 = 0 < t_1 < \cdots < t_M = T$, with step size $\delta = T/M$ (so $t_k = k\delta$). To compute the M American option prices $V(e^{\gamma}, t_M)$, $V(e^{\gamma}, t_{M-1})$, $\ldots$, $V(e^{\gamma}, t_1)$ with a single run of the backward induction program, we approximate the Brownian motion Z_t (with drift λ) by a Bernoulli random walk with time increment $\delta > 0$ and space increment X_i such that

$$P\{X_i = \pm\sqrt{\delta}(\sigma^2 + \delta\lambda^2)^{\frac{1}{2}}\} = \frac{1}{2}\left(1 \pm \frac{\lambda\sqrt{\delta}}{\sqrt{\sigma^2 + \delta\lambda^2}}\right) \tag{11}$$

Note that X_i has mean $\lambda\delta = E(Z_{t+\delta} - Z_t)$ and variance $\sigma^2\delta = \text{Var}(Z_{t+\delta} - Z_t)$, and that the barrier γ belongs to the lattice $L_\delta = \{\gamma \pm \sqrt{\delta}\,(\sigma^2 + \delta\lambda^2)^{1/2} j: j = 0, 1, 2, \ldots\}$. The backward induction algorithm of dynamic programming yields:

$$V(e^x, t_{k-1}) = \max\{(K - e^x)^+,\ e^{-r\delta} E V(e^{x+X_k}, t_k)\} \quad \text{for } x \in L_\delta \tag{12}$$

and is initialized at T by $V(e^x, T) = (K - e^x)^+$.

Note that $\log S$ may not belong to the lattice L_δ, in contrast to the usual binomial tree method in which S is always the root node of the tree but the barrier may not be a node of the tree. In AitSahlia, Imhof, and Lai [2003], we use a similar Bernoulli random walk with absorbing barrier γ and increments (11) to approximate a Wiener process with the same absorbing barrier γ to handle the barrier problem for knock-out options (see Boyle and Lau [1994]).

III. FAST AND ACCURATE APPROXIMATION

Since the price $V(H, t)$ of a standard American put option can be decomposed as the sum of a European put plus an early exercise premium, we can likewise decompose the price (6) of an American down-and-in put as:

$$P_D(S) = p_D(S) + \int_0^T e^{-rt} \pi(H, T-t) f_z(t)\, dt \tag{13}$$

where $\pi(H, T - t)$ is the early exercise premium of a standard American put with maturity $T - t$, strike price K, and initial stock price H; $f_z(t)$ is given in (9); and $z = \log S$.

The $p_D(S)$ in (13) is the price of a European down-and-in put option with strike price K and expiration date T. In view of (1), $p_D(S)$ can be expressed as the difference between a standard European put and a European down-and-out put, yielding the closed-form expression:

$$
\begin{aligned}
p_D(S) &= -Se^{-qT}N(-d_1(S,H,T)) + Ke^{-rT}N(-d_2(S,H,T)) \\
&\quad + Se^{-qT}(H/S)^{2\lambda+2}\{N(d_1(H^2,SK,T)) - N(d_1(H,S,T))\} \\
&\quad - Ke^{-rT}(H/S)^{2\lambda}\{N(d_2(H^2,SK,T)) - N(d_2(H,S,T))\}
\end{aligned}
\tag{14}
$$

where $d_1(x, y, \tau) = \{\log(x/y) + (r - q + \sigma^2/2)\tau\} / (\sigma\sqrt{\tau})$; $d_2(x, y, \tau) = d_1(x, y, \tau) - \sigma\sqrt{\tau}$; and $N(x)$ is the standard normal cumulative probability distribution function.

At this point, the early exercise premium $\pi(H, T - t)$ is the only piece needed to fully determine the price of an American down-and-in put using Equation (13). Since this quantity is the difference between standard European and American option prices, for which various numerical methods are available, one may choose a method according to one's preference. Nevertheless, once π is determined, there remains the task of evaluating the integral in (13).

For fast and accurate approximations of the integrand and integral in (13), first we use Ju's [1998] or AitSahlia and Lai's [2001] method to approximate the early exercise premium π. The method involves approximating the early exercise boundary by a piecewise exponential function (Ju) in the original geometric Brownian motion scale or by a piecewise linear function (AitSahlia and Lai) in the Brownian motion scale resulting from a change of variables. A major advantage of this approach is that it leads to closed-form approximations of the early exercise premium.

Multiplying the discounted American option premium $e^{-rt}\pi(H, T - t)$ by the first-passage density $f_z(t)$ then gives the value of the integrand in (13). For fast valuation, the integral has to be approximated by a sum of m terms with small m. We evaluate it using Gaussian quadrature with m nodes and weight function f_z, which results in a weighted sum that approximates the integral in (13) (see Press et al. [1992, Section 4.5]).

The weights $w_1, \ldots, w_m$ are determined together with nodes $t_1, \ldots, t_m$ so that

$$
\int_0^T h(t) f_z(t)\, dt = \sum_{k=1}^{m} w_k h(t_k)
$$

for all polynomials h of degree less than $2m$.

To find these nodes and weights, a first step is to evaluate the moments

$$
c_k = \int_0^T t^k f_z(t)\, dt \text{ for } k = 0, \ldots, 2m-1
$$

Note that c_0 is exactly the probability (under the risk-neutral measure) that the barrier is hit (i.e., knock-in occurs) during the life of the option. Let $a = z - y$ and $b = -\lambda$. Recalling that $n(x)$ denotes the standard normal density and $N(x)$ its cumulative distribution function, we have:

$$
\begin{aligned}
\frac{c_0}{2} + \frac{bc_1}{2a} &= \int_0^T \left(\frac{at^{-3/2}}{2\sigma} + \frac{bt^{-1/2}}{2\sigma} \right) n\left(\frac{at^{-1/2}}{\sigma} - \frac{bt^{1/2}}{\sigma} \right) dt \\
&= N_1 - N_2
\end{aligned}
\tag{15}
$$

where

$$
\begin{aligned}
N_1 &= 1 \\
N_2 &= N\left(\frac{a}{\sigma\sqrt{T}} - \frac{b\sqrt{T}}{\sigma} \right)
\end{aligned}
$$

Similarly:

$$
\frac{c_0}{2} - \frac{bc_1}{2a} = \exp\left(\frac{2ab}{\sigma^2} \right)(M_1 - M_2) \tag{16}
$$

where $M_1 = 1$ and $M_2 = N(a/(\sigma\sqrt{T}) + b\sqrt{T}/\sigma)$. From (15) and (16):

$$
c_0 = N_1 - N_2 + \exp\left(\frac{2ab}{\sigma^2} \right)(M_1 - M_2) \tag{17}
$$

and $c_1 = (a/b)\{2(N_1 - N_2) - c_0\}$ if $b \neq 0$.

To compute the higher moments, note that by partial integration:

$$
c_k = \beta_k - \frac{a^2}{(2k-1)\sigma^2} c_{k-1} + \frac{b^2}{(2k-1)\sigma^2} c_{k+1}
$$

where

$$
\beta_k = \frac{a}{\sigma(k - \frac{1}{2})} T^{k-\frac{1}{2}} n\left(\frac{a - bT}{\sigma\sqrt{T}} \right)
$$

Therefore, in the case $b \neq 0$:

$$
c_k = \frac{(2k-3)\sigma^2}{b^2}(c_{k-1} - \beta_{k-1}) + \frac{a^2}{b^2} c_{k-2}
$$

If $b = 0$, then:

$$c_k = \beta_k - \frac{a^2}{(2k-1)\sigma^2} c_{k-1}$$

The moments $c_0, \ldots, c_{2m-1}$ can therefore be evaluated recursively. Let

$$q_\nu(t) = \det \begin{pmatrix} c_0 & c_1 & \cdots & c_{\nu-1} & 1 \\ c_1 & c_2 & \cdots & c_\nu & t \\ \cdots & \cdots & & \cdots & \cdots \\ c_\nu & c_{\nu+1} & \cdots & c_{2\nu-1} & t^\nu \end{pmatrix}$$

$$D = \det \begin{pmatrix} c_0 & c_1 & \cdots & c_{m-1} \\ c_1 & c_2 & \cdots & c_m \\ \cdots & \cdots & & \cdots \\ c_{m-1} & c_m & \cdots & c_{2m-2} \end{pmatrix}$$

Note that the polynomials $q_0(t), \ldots, q_m(t)$ are orthogonal with respect to $f_z(t)dt$ in $[0, T]$. The nodes $t_1, \ldots, t_m$ are the zeros of $q_m(t)$, and the weights $w_1, \ldots, w_m$ are given by

$$w_k = \frac{D^2}{q'_m(t_k)q_{m-1}(t_k)} \quad \text{for } k = 1, \ldots, m$$

Finally, to evaluate the integral in (13) we use the approximation:

$$\int_0^T e^{-rt}\pi(H, T-t)f_z(t)\,dt \approx \sum_{k=1}^{m} w_k e^{-rt_k}\pi(H, T-t_k) \quad (18)$$

choosing m to be small for fast computation.

IV. NUMERICAL ILLUSTRATION

Our numerical examples for the two methods consider both a short-maturity put option on a security paying no dividend (*Exhibit 1*) and a long-maturity put option on a security paying a dividend at a constant rate (*Exhibit 2*). The middle four columns are generated by the modified binomial algorithm in Equations (10) and (12), for values of N equal to 1,000, 5,000, 10,000 and 20,000. The expected convergence of the algorithm is clearly noticeable.

The last columns in the Exhibits present the integral approximation method with $m = 2$. In this case, the price of the American knock-in put is obtained via the approximation (18) of the integral in (13).

Observe that with $N = 1{,}000$, the modified binomial algorithm generally yields penny accuracy, and that the integral approximation with $m = 2$ nodes, which is over ten times faster, is even more accurate. The integral approximation involves c_0 given in (17), which is equal to the probability of ever hitting the barrier during the life of the option, also tabulated in Exhibits 1 and 2.

Note that one could improve the accuracy of the integral approximation by increasing the number m of nodes, thus resulting in higher-degree polynomials $q_m(t)$.

EXHIBIT 1

American Down-and-In Put Option Prices: Modified Binomial Algorithm and Integral Approximation (with m = 2)

S	H	Prob. of Hitting Barrier	Modified Binomial $N = 1000$	$N = 5000$	$N = 10000$	$N = 20000$	Integral Approx.
75	70	0.5821	17.3026	17.3007	17.3005	17.3004	17.3003
80	70	0.3004	8.8797	8.8773	8.8769	8.8768	8.8767
85	70	0.1389	4.0931	4.0911	4.0909	4.0907	4.0906
90	70	0.0584	1.7153	1.7139	1.7138	1.7137	1.7136
85	80	0.6271	12.4374	12.4362	12.4361	12.4360	12.4346
90	80	0.3583	7.0680	7.0664	7.0662	7.0661	7.0649
95	80	0.1878	3.6918	3.6903	3.6901	3.6900	3.6894
100	80	0.0911	1.7862	1.7851	1.7850	1.7849	1.7847
95	90	0.6636	6.8017	6.8011	6.8010	6.8010	6.8050
100	90	0.4089	4.1201	4.1193	4.1192	4.1178	4.1244
105	90	0.2350	2.3422	2.3414	2.3413	2.3413	2.3455
110	90	0.1268	1.2542	1.2536	1.2535	1.2532	1.2557
Av. CPU time (sec.)			0.15	3.60	14.24	58.51	< 0.01

$r = 0.06$, $q = 0$, $\sigma = 0.2$, $T = 0.5$, $K = 100$.

EXHIBIT 2

American Down-and-In Put Option Prices: Modified Binomial Algorithm and Integral Approximation (with m = 2)

S	H	Prob. of Hitting Barrier	Modified Binomial $N = 1000$	$N = 5000$	$N = 10000$	$N = 20000$	Integral Approx.
80	70	0.8074	24.4783	24.4768	24.4766	24.4765	24.4760
90	70	0.6161	17.8960	17.8933	17.8930	17.8928	17.8922
100	70	0.4505	12.7056	12.7022	12.7018	12.7016	12.7009
110	70	0.3190	8.8116	8.8081	8.8076	8.8074	8.8070
120	70	0.2206	6.0021	5.9988	5.9984	5.9982	5.9978
90	80	0.8320	19.5239	19.5226	19.5224	19.5223	19.5217
100	80	0.6622	14.6116	14.6097	14.6094	14.6093	14.6090
110	80	0.5088	10.7227	10.7204	10.7201	10.7199	10.7198
120	80	0.3804	7.7411	7.7386	7.7383	7.7381	7.7381
100	90	0.8512	15.0884	15.0871	15.0869	15.0868	15.0861
110	90	0.6990	11.3287	11.3272	11.3270	11.3268	11.3266
120	90	0.5574	8.3962	8.3946	8.3944	8.3942	8.3942
Av. CPU time (sec.)			0.13	3.55	15.41	68.37	< 0.01

$r = 0.06$, $q = 0.09$, $\sigma = 0.2$, $T = 3$, $K = 100$.

Alternatively, one could achieve the same goal by dividing the interval [0, T] into several subintervals of the same length, and then applying Gaussian quadrature to each subinterval, but with corresponding orthogonal polynomials of degree 2, resulting in similar formulas.

Both approaches will converge to the correct value of the integral as the number of nodes (or subintervals) tends to infinity. The advantage of the second approach is that the nodes in each subinterval are easily calculated as they are zeros of a quadratic function. For the first approach, one must numerically determine all the zeros of an mth-degree polynomial. Our numerical results suggest that using only two nodes for the entire interval gives very accurate results in the first place.

V. AMERICAN KNOCK-IN HEDGE PARAMETERS

From (6), it follows that the hedge parameter Δ is given by:

$$\Delta = \frac{\partial P_D}{\partial S}(S) = \int_0^T e^{-rt} V(H,t) \frac{\partial f_z}{\partial S}(t) dt \tag{19}$$

Since $z = \log S$, the chain rule yields:

$$\frac{\partial f_z}{\partial S}(t) = \frac{1}{S}\frac{\partial f_z}{\partial z}(t)$$

Moreover, from (9), it follows that:

$$\frac{\partial f_z}{\partial z}(t) = \frac{1}{\sigma t^{3/2}} \left\{ 1 + (z-\gamma)\left(\frac{\gamma - z - \lambda t}{\sigma^2 t}\right) \right\} n \left(\frac{\gamma - z - \lambda t}{\sigma\sqrt{t}}\right)$$

Therefore, as in (10), we can evaluate Δ via:

$$\Delta = \frac{\delta}{S}\sum_{k=1}^{M} e^{-rk\delta} V\left(e^{\gamma}, k\delta\right) \frac{\partial f_z}{\partial z}(k\delta) \tag{20}$$

Similar expressions can be obtained for the hedge parameters gamma and theta. Note that (20) shows that the Bernoulli walk algorithm to compute Δ does not involve numerical differentiation. Alternatively, one can use the decomposition formula (13) to compute the hedge parameters. For example:

$$\Delta = \frac{\partial p_D}{\partial S}(S) + \int_0^T e^{-rt}\pi(H, T-t)\frac{\partial f_z}{\partial S}(t)dt \tag{21}$$

Here $\partial p_D/\partial S$ is given in closed form because of (14), and so is $\partial f_z/\partial S$. To evaluate the integral in (21), we can also use Gaussian quadrature.

VI. SUMMARY

We have considered American knock-in options, for which the integral defining the early exercise premium is very different from that of an American knock-out option. Despite the non-Markovian nature of the associated optimal stopping problem, we have been able to develop a modified binomial algorithm to price American knock-in options. We have also given an alternative approach that makes use of the classic decomposition formula for a standard American option and computes the early exercise premium by Gaussian quadrature.

The methods presented for American down-and-in put options can be modified for up-and-in put options and for the corresponding call options.

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