Keyword Search Advertising and Limited Budgets

Online Appendix

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In this appendix, we formally prove the claims made in the main text and provide the analysis of the extension section as well as the proofs for Proposition 5 and Proposition 6.

**Proof of Claims**

In this section, we provide proofs for the claims made in the main text.

**Claim A1.** When \((1 - \delta)r_{LV} < K_H < \delta r_M v_M + (1 - \delta)r_{LV} < \min\{K_M, K_L\}\), \(M-H-L\) can be an equilibrium listing order if \(K_H\) is small enough.

**Proof.** Suppose the equilibrium listing order is \(M-H-L\). Then the bids satisfy the following IC conditions:

\[
\Pi_{M1} = r_M v_M - r_H b_H \geq \Pi_{M2} = \left(\frac{K_H}{r_H v_H}\right)(1-\delta)\left(1+\frac{K_H}{r_H v_H}\right)\left(r_M v_M - r_L b_L\right) \tag{A1}
\]

\[
\Pi_{H2} = (1-\delta)(r_H v_H - r_L b_L) \geq \Pi_{H1} = \frac{K_H}{r_H v_H}r_H v_H - K_H \tag{A2}
\]

\[
\Pi_{H2} = (1-\delta)(r_H v_H - r_L b_L) \geq \Pi_{H3} = 0 \tag{A3}
\]

\[
\Pi_{L3} = 0 \geq \Pi_{L2} = (1-\delta)(r_L v_L - r_L b_L) \tag{A4}
\]

It is easy to see that (A1) and (A2) respectively define the upper bound and the lower bound of \(b_H\), while (A3) and (A4) respectively determine the upper bound and the lower bound of \(b_L\). Note that \(\frac{\partial \Pi_{L3}}{\partial b_L} = 0\). In addition, since by (A3) and (A4), \(\frac{\partial \Pi_{L2}}{\partial b_L} = 0\) at any value of \(b_L\) in the equilibrium interval, we have \(\frac{\partial \Pi_{L2}}{\partial b_L} = \frac{\partial \Pi_{L3}}{\partial b_L} + \frac{\partial \Pi_{L2}}{\partial b_L} \frac{\partial b_L}{\partial b_L} = 0\). Thus, any combination of bids satisfying (A1)-(A4) as well as \(r_M b_M \geq r_H b_H\) can be a Symmetric Nash Equilibrium. Thus, the Pareto-dominant equilibrium is given by the lower bound solutions:

\[
b^*_L = v_L \tag{A5}
\]

\[
b^*_H = \frac{K_H v_H}{K_H + (1-\delta)(r_H v_H - r_L v_L)} \tag{A6}
\]

\[
b^*_M = \frac{K_H r_H v_H}{r_M (K_H + (1-\delta)(r_H v_H - r_L v_L))} \tag{A7}
\]

Then the condition for \(M-H-L\) to be an equilibrium listing order can be obtained by plugging these solutions in (A1), as follows:

\[
K_H \leq \frac{(r_H v_H - r_L v_L)(r_M v_M - r_L v_L) - r_L v_H(r_H v_H - r_L v_L)}{2\delta(r_M v_M - r_L v_L)} \tag{A8}
\]

\[\square\]

**Claim A2.** When \(\min\{K_H, K_L\} \geq \delta r_M v_M + (1 - \delta)r_{LV} \) and \((1 - \delta)C_0 < K_M < (1 - \delta)r_{LV}\), the equilibrium listing order is always \(H-M-L\).

**Proof.** The condition implies that only Firm M has a binding budget constraint at the second slot but that Firm H and Firm L are not budget-constrained. In this case, by the result of Section 3.1, Firm H should always be listed higher than Firm L. Thus, only the following
three listing orders can be observed in equilibrium: H-M-L, H-L-M, and M-H-L. Given that the main text already showed that H-M-L is always an equilibrium listing order, we prove neither H-L-M nor M-H-L can be an equilibrium listing order.

First, suppose the equilibrium listing order is given by H-L-M. Then Firm L should prefer winning the second slot to losing but Firm M should prefer losing to winning the second slot. These imply,

\[ \Pi_{L2} = (1 - \delta)(r_L v_L - r_M b_M) \geq \Pi_{L3} = \left(1 - \frac{K_M}{(1 - \delta)^2 v_M} \right)(1 - \delta)(r_L v_L - C_0) \quad (A9) \]

\[ \Pi_{M3} = 0 \geq \Pi_{M2} = \left(1 - \frac{K_M}{(1 - \delta)^2 v_M} \right)(1 - \delta)r_M v_M - K_M. \quad (A10) \]

However, both (A9) and (A10) cannot simultaneously hold if \( K_M < (1 - \delta)r_L v_L \), because (A10) implies \( b_M \geq v_M \), which in turn implies \( \Pi_{L2} \leq 0 \) but this contradicts (A9). Therefore, H-L-M cannot be an equilibrium listing order.

Finally, suppose the equilibrium listing order is given by M-H-L. Then Firm M should prefer the first slot to the second, while Firm H should prefer the second slot to the first. These imply,

\[ \Pi_{M1} = \left(1 - \frac{K_M}{r_H b_H} \right)r_M v_M - K_M \geq \Pi_{M2} = \left(1 - \frac{K_M}{(1 - \delta)^2 v_M} \right)(1 - \delta)r_M v_M - K_M \quad (A11) \]

\[ \Pi_{H2} = \left\{ \left(1 - \frac{K_M}{r_H b_H} \right)(1 - \delta) + \left(1 - \frac{K_M}{r_H b_H} \right) \right\} \geq \Pi_{H1} = (1 - \delta)r_H v_H - \left(1 - \frac{K_M}{r_L b_L} \right)r_L b_L. \quad (A12) \]

Therefore, no other listing order than H-M-L can be observed in equilibrium. \[ \square \]

**Claim A3.** When \( (1 - \delta)C_0 < K_M < (1 - \delta)r_L v_L < K_H < \delta r_M v_M + (1 - \delta)r_L v_L < K_L \), only H-M-L and H-L-M can be equilibrium listing orders.

**Proof.** The conditions imply that Firm H has a binding budget constraint at the first slot and that Firm M has a binding budget constraint at both the first and the second slots. To prove the claim, we first show that M-H-L, M-L-H, L-H-M, and L-M-H cannot be observed in equilibrium (Part 1) and then show the existence of an equilibrium with the rest of the orders (Part 2).

**Part 1.** Suppose the equilibrium listing order is M-H-L. In this case, since Firm H is not budget-constrained in the second slot, Firm M’s IC condition is identical to (A11). Then as in the above proof, we have \( r_H b_H = r_L b_L \). However, this implies that the following Firm H’s IC condition:

\[ \Pi_{H2} = \left\{ \left(1 - \frac{K_M}{r_H b_H} \right)(1 - \delta) + \left(1 - \frac{K_M}{r_H b_H} \right) \right\} \geq \Pi_{H1} = \left(1 - \frac{K_M}{r_H b_H} \right)(r_H v_H - r_H b_H), \quad (A13) \]

is equivalent to,

\[ \left\{ \left(1 - \frac{K_M}{r_H b_H} \right)(1 - \delta) + \left(1 - \frac{K_M}{r_H b_H} \right) \right\} r_L b_L \geq K_H, \quad (A14) \]
which contradicts the fact that Firm H is not budget-constrained at the second slot. Thus, there is no such equilibrium.

Suppose the equilibrium listing order is M-L-H. Under this listing order, only Firm M has a binding budget constraint. Then the IC conditions of Firm L and Firm H are:

\[ \Pi_{L2} = \left( \frac{K_M}{r_M} \right) (1-\delta) \geq \Pi_{L3} = \left( \frac{K_M}{r_M} \right) (1-\delta)(r_{L}v_{L} - C_0) \quad \text{(A15)} \]

\[ \Pi_{H2} = \left( \frac{K_H}{r_H} \right) (1-\delta)(r_{H}v_{H} - C_0) \geq \Pi_{L2} = \left( \frac{K_M}{r_M} \right) (1-\delta)(r_{L}v_{L} - C_0) \quad \text{(A16)} \]

which imply \( r_{L}v_{L} \geq r_{H}v_{H} \). This is a contradiction and thus, there is no such equilibrium.

Now, suppose the equilibrium listing order is L-H-M. Under this listing order, no advertiser has a binding budget constraint. This implies \( r_{L}v_{L} > r_{H}v_{H} > r_{M}v_{M} \) by the result of Section 3.1, which is a contradiction. Thus, there is no equilibrium with the listing order L-H-M.

Finally, suppose the equilibrium listing order is L-M-H. Under this listing order, Firm M has a binding budget constraint at the second slot. Then since Firm L prefers the first rank to the third while Firm H prefers the third rank to the first, we have

\[ \Pi_{L1} = r_{L}v_{L} - \left( \frac{K_M}{(1-\delta)r_{H}b_{H}} \right) r_{M}b_{M} - \left( \frac{K_M}{(1-\delta)r_{M}b_{M}} \right) r_{H}b_{H} \geq 0 \]

\[ \Pi_{H3} = \left( \frac{K_H}{(1-\delta)r_{M}b_{M}} \right) (1-\delta)(r_{H}v_{H} - C_0) \geq \Pi_{L1} = r_{L}v_{L} - \left( \frac{K_M}{(1-\delta)r_{H}b_{H}} \right) r_{M}b_{M} - \left( \frac{K_M}{(1-\delta)r_{M}b_{M}} \right) r_{H}b_{H} \]

which imply \( r_{L}v_{L} \geq r_{H}v_{H} \). This is a contradiction and thus, there is no equilibrium with the listing order L-M-H.

\textbf{Part 2.} We now show the existence of an equilibrium with the listing orders H-M-L and H-L-M, by giving a numerical example. First suppose that the equilibrium listing order is H-M-L. In this case, if \( r_{H} = r_{M} = r_{L} = 1 \), \( v_{H} = 25 \), \( v_{M} = 11 \), \( v_{L} = 10 \), \( \delta = 0.5 \), \( C_0 = 1 \), \( K_{H} = 9 \), \( K_{M} = 4.5 \), and \( K_{L} = 15 \), then the equilibrium bids are given as \( b_{M} = 10.58 \) and \( b_{L} = 9.43 \) from simultaneously solving (35) and (36). Plugging these solutions back into the IC conditions (32) and (34), we confirm that both are satisfied: \( \Pi_{M2} = 0.75 > \Pi_{M1} = 0.18 \) and \( \Pi_{L3} = 1.11 > \Pi_{L2} = 0.33 \). Finally, the budgets satisfy the assumptions: \( (1-\delta)C_0(=0.5) < K_{M}(=4.5) < (1-\delta)r_{L}v_{L}(=5) < K_{H}(=9) < \delta r_{M}v_{M} + (1-\delta)r_{L}v_{L}(=10.5) < K_{L}(=15) \), and \( \frac{K_{H}}{r_{M}b_{M}} (=0.85) < \frac{K_{M}}{r_{L}b_{L}} (=0.95) \). Thus, there exists an equilibrium with the listing order H-M-L.

Now, suppose the equilibrium listing order is H-L-M. Further, suppose Firm H is more budget-constrained than Firm M when Firm M deviates to the second slot. Then the IC conditions are given as follows:

\[ \Pi_{H1} = \left( \frac{K_{H}}{r_{L}b_{L}} \right) r_{H}v_{H} - K_{H} \geq 0 \]

\[ \Pi_{L2} = \left( \frac{K_{H}}{r_{H}b_{H}} \right) (1-\delta) + \left( \frac{K_{M}}{r_{H}b_{H}} \right) r_{L}v_{L} - r_{M}b_{M} \geq r_{L}v_{L} - r_{L}b_{L} \]

\[ \Pi_{L2} = \left( \frac{K_{H}}{r_{L}b_{L}} \right) (1-\delta) + \left( \frac{K_{M}}{r_{L}b_{L}} \right) r_{L}v_{L} - r_{M}b_{M} \geq \Pi_{L1} = r_{L}v_{L} - r_{L}b_{L} \]

\[ \Pi_{H3} = \left( \frac{K_{M}}{r_{L}b_{L}} \right) (1-\delta)(r_{H}v_{H} - C_0) \geq \Pi_{M3} = \left( \frac{K_{M}}{r_{L}b_{L}} \right) (1-\delta)r_{M}v_{M} - K_{M} \]

\[ \Pi_{M2} = \left( \frac{K_{M}}{r_{L}b_{L}} \right) r_{M}v_{M} - K_{M} \]
Now to show the existence of the equilibrium, it suffices to find a case where all the IC conditions in (A19)-(A22) hold. Suppose \( r_H = r_M = r_L = 1 \), \( v_H = 35 \), \( v_M = 12 \), \( v_L = 11 \), \( \delta = 0.6 \), and \( C_0 = 1 \). If we let \( K_H = 5.5 \), \( K_M = 1 \), and \( K_L = 15 \), then we obtain \( b_L = 10.81 \) and \( b_M = 4.22 \) by simultaneously solving (A19) and (A21).\(^1\) Plugging these solutions back into (A20) and (A22), we have \( \Pi_{L2} = 4.71 > \Pi_{L1} = 0.19 \) and \( \Pi_{M3} = 2.16 > \Pi_{M2} = 1.84 \) respectively. Thus, all the IC conditions can be satisfied in this case. Finally, the assumptions are satisfied as well: \( r \) conditions in (A19)-(A22) hold. Suppose \( \Pi \) listing order is H-M. The IC conditions are given as, \( \Pi_{H1} = r_H v_H - r_M b_M \geq \Pi_{H2} = (1 - \delta)(r_H v_H - C_1) \) (A23) \( \Pi_{M2} = (1 - \delta)(r_M v_M - C_1) \geq \Pi_{M1} = r_M v_M - r_M b_M \). (A24) Then Firm M’s bid in the Pareto-dominant equilibrium is given as \( b^*_M = \delta v_M + (1 - \delta)\frac{C_1}{r_M} \).

**Proof of Proposition 5**

First note that in the analysis of the extensions, we consider only two advertisers. Thus, it is sufficient to check the partial derivative in confirming the existence of the bid-raising incentive. To prove Proposition 5, we first derive the equilibrium of a one-shot game in the following lemma.

**Lemma A1.** Suppose Firm L’s bid is exogenously given and let \( C_1 = r_L b_L \). Further suppose the budget constraints are never binding at the second slot: \( K_i \geq (1 - \delta)C_1 \) \( i = H, M, L \). Then in the equilibrium of a one-shot bidding game, Firm H always takes the first slot when its budget constraint is not binding at the first slot but both firms can take the first slot when Firm H’s budget constraint is binding at the first slot. The specific conditions and the equilibrium solutions are given in the proof.

**Proof.** We consider the following four cases: (1) when no firm has a binding budget constraint: \( K_i \geq \delta r_M v_M + (1 - \delta)C_1 \) (denote this case by NN), (2) when only Firm H has a binding budget constraint: \( K_H < \delta r_M v_M + (1 - \delta)C_1 \leq K_M \) (denote it by NB), (3) when only Firm M has a binding budget constraint: \( K_M < \delta r_M v_M + (1 - \delta)C_1 \leq K_H \) (denote it by BN), and (4) when both firms are budget-constrained: \( K_i < \delta r_M v_M + (1 - \delta)C_1 \) (denote it by BB). We derive the bidding equilibrium in four parts.

**Part 1.** First consider the NN case. The result of Section 3.1 still holds here and thus the equilibrium listing order is H-M. The IC conditions are given as,

\[
\Pi_{H1} = r_H v_H - r_M b_M \geq \Pi_{H2} = (1 - \delta)(r_H v_H - C_1)
\]

\[
\Pi_{M2} = (1 - \delta)(r_M v_M - C_1) \geq \Pi_{M1} = r_M v_M - r_M b_M.
\]

\(^1\)In fact, it can be easily shown that \( \frac{d\Pi_{L2}}{db_L} > 0 \) and that \( \frac{d\Pi_{L2}}{db_M} > 0 \). Thus, the upper bound solutions defined by (A19) and (A21) are indeed equilibrium solutions.
Based on this, Firm H’s total advertising cost at the first slot is given from (A24) as,

\[ C_H = \delta r_M v_M + (1 - \delta) C_1. \]  

(A25)

Thus, the equilibrium profits are given as,

\[ \Pi_H = r_H v_H - \{\delta r_M v_M + (1 - \delta) C_1\} \]  

(A26)

\[ \Pi_M = (1 - \delta)(r_M v_M - C_1). \]  

(A27)

**Part 2.** Consider the BN case. Since Firm H’s budget constraint is binding at the first slot, the listing orders may be either H-M or M-H. Thus, we consider these two listing orders.

First, suppose the equilibrium listing order is H-M. Then the IC conditions are given as,

\[ \Pi_{H1} = \left( \frac{K_H}{r_M v_M} \right) r_H v_H - K_H \geq \Pi_{H2} = (1 - \delta)(r_H v_H - C_1) \]  

(A28)

\[ \Pi_{M2} = \left\{ \left( \frac{K_H}{r_M v_M} \right) (1 - \delta) + \left( 1 - \frac{K_H}{r_M v_M} \right) \right\} (r_M v_M - C_1) \geq \Pi_{M1} = r_M v_M - r_M b_M \]  

(A29)

Since

\[ \frac{\partial \Pi_{M2}}{\partial b_M} = \left( \frac{\delta K_H}{r_M v_M} \right) (r_M v_M - C_1) > 0, \]  

(A30)

Firm M’s optimal bid amount is determined at the upper bound, by (A28). Thus, Firm H’s total advertising cost in equilibrium is given by,

\[ C_H = r_M b_M^* = \frac{K_H r_H v_H}{K_H + (1 - \delta)(r_H v_H - C_1)}. \]  

(A31)

Corresponding profits are given as,

\[ \Pi_H = (1 - \delta)(r_H v_H - C_1) \]  

(A32)

\[ \Pi_M = \frac{(r_M v_M - C_1)(r_H v_H - \delta K_H - \delta(1 - \delta)(r_H v_H - C_1))}{r_H v_H}. \]  

(A33)

By plugging in (A31) into (A29) and solving for \( K_H \), we have the condition for H-M to be an equilibrium listing order, given as follows:

\[ K_H > K_H^{(1)} = \frac{(r_H v_H - C_1)(r_H v_H - 2(1 - \delta)(r_M v_M - C_1))}{2\delta(r_M v_M - C_1)}. \]  

(A34)

Next suppose the equilibrium listing order is M-H. Then the bids satisfy the following IC conditions:

\[ \Pi_{M1} = r_M v_M - r_H b_H \geq \Pi_{M2} = \left\{ \left( \frac{K_H}{r_H v_H} \right) (1 - \delta) + \left( 1 - \frac{K_H}{r_H v_H} \right) \right\} (r_M v_M - C_1) \]  

(A35)

\[ \Pi_{H2} = (1 - \delta)(r_H v_H - C_1) \geq \Pi_{H1} = \left( \frac{K_H}{r_H v_H} \right) r_H v_H - K_H \]  

(A36)
Then (A36) defines the Pareto-dominant equilibrium and based on this, we derive the total advertising cost of Firm M as well as the profits of both firms as follows:

\[ C_M = \frac{K_H r_H v_H}{K_H + (1-\delta)(r_H v_H - C_1)} \]  
(A37)

\[ \Pi_H = (1-\delta)(r_H v_H - C_1) \]  
(A38)

\[ \Pi_M = r_M v_M - \frac{K_M r_M v_M}{K_H + (1-\delta)(r_H v_H - C_1)} \]  
(A39)

By plugging the equilibrium solution in (A5), the condition for M-H to be an equilibrium listing order is given as follows: \( K_H \leq K^{(1)}_M \), where \( K^{(1)}_M \) is defined in (A44).

Part 3. Consider the NB case. We also consider both listing orders here: H-M and M-H.

First, suppose the equilibrium listing order is H-M. Then the IC conditions are given as:

\[ \Pi_{H1} = r_H v_H - r_M b_M \geq \Pi_{H2} = \left\{ \frac{K_M}{r_M b_M} \right\} (1-\delta) + \left(1 - \frac{K_M}{r_M b_M} \right) (r_H v_H - C_1) \]  
(A40)

\[ \Pi_{M2} = (1-\delta)(r_M v_M - C_1) \geq \Pi_{M1} = \left( \frac{K_M}{r_M b_M} \right) r_M v_M - K_M. \]  
(A41)

Since Firm M’s bid in the Pareto-dominant equilibrium is given as \( b_M^* = \frac{K_M r_M v_M}{r_M (K_M + (1-\delta)(r_M v_M - C_1))} \), Firm M’s total advertising cost at the first slot is, by (A41),

\[ C_H = r_M b_M^* = \frac{K_M r_M v_M}{K_M + (1-\delta)(r_M v_M - C_1)}. \]  
(A42)

Then the profits are

\[ \Pi_H = r_H v_H - \frac{K_M r_M v_M}{K_M + (1-\delta)(r_M v_M - C_1)} \]  
(A43)

\[ \Pi_M = (1-\delta)(r_M v_M - C_1). \]  
(A44)

For H-M to be an equilibrium listing order, (A40) should hold when the equilibrium solution is plugged in, that is, \( K_M \leq K^{(1)}_M \) or \( K_M \geq K^{(2)}_M \), where

\[ K^{(1)}_M = \frac{(r_M v_M - C_1)(r_M v_M - 2\delta(1-\delta)(r_H v_H - C_1) - r_M v_M \sqrt{(r_M v_M - C_1)(r_M v_M - C_1) - 4\delta(1-\delta)(r_H v_H - C_1)}}{2\delta(r_H v_H - C_1)} \]  
(A45)

\[ K^{(2)}_M = \frac{(r_M v_M - C_1)(r_M v_M - 2(1-\delta)(r_H v_H - C_1) + r_M v_M \sqrt{(r_M v_M - C_1)(r_M v_M - C_1) - 4(1-\delta)(r_H v_H - C_1)}}{2(1-\delta)(r_H v_H - C_1)} \]  
(A46)

Note that if \( r_M v_M - C_1 \leq 4\delta(1-\delta)(r_H v_H - C_1) \), \( K^{(1)}_M \) and \( K^{(2)}_M \) are not defined and H-M is always an equilibrium listing order. Also note that if \( r_M v_M - C_1 > 4\delta(r_H v_H - C_1) \), both \( K^{(1)}_M \) and \( K^{(2)}_M \) are defined but since \( K^{(2)}_M > \delta r_M v_M + (1-\delta)C_1 \), the listing order is given H-M if and only if \( K_M \leq K^{(1)}_M \) in the specified range for the NB case.

Next suppose the listing order is M-H. Then IC conditions are given as,

\[ \Pi_{M1} = \left( \frac{K_M}{r_H b_M} \right) r_M v_M - K_M \geq \Pi_{M2} = (1-\delta)(r_M v_M - C_1) \]  
(A47)

\[ \Pi_{H2} = \left\{ \frac{K_M}{r_H b_M} \right\} (1-\delta) + \left(1 - \frac{K_M}{r_H b_M} \right) (r_H v_H - C_1) \geq \Pi_{H1} = r_H v_H - r_H b_H. \]  
(A48)

Since

\[ \frac{\partial \Pi_{H2}}{\partial \delta} = \left( \frac{\delta K_M}{r_H b_M} \right) (r_H v_H - C_1) > 0, \]  
(A49)
Firm H’s optimal bid amount is determined at the upper bound and thus Firm H’s total advertising cost is given from (A47) by:

$$C_M = r_H b_H^* = \frac{K_M r_{MvM}}{K_M + (1-\delta)(r_{MvM} - C_1)}.$$  \hfill (A50)

Corresponding profits are given as,

$$\Pi_H = \frac{(r_H v_H - C_1)(r_{MvM} - \delta K_M - \delta(1-\delta)(r_{MvM} - C_1))}{r_{MvM}}$$  \hfill (A51)

$$\Pi_M = (1-\delta)(r_{MvM} - C_1).$$  \hfill (A52)

By plugging the equilibrium bid into (A48) and solving for \(K_M\), the condition for M-H to be an equilibrium listing order is given as \(K_M \geq K_M^{(1)}\) if \(r_{MvM} - C_1 > 4\delta(1-\delta)(r_H v_H - C_1)\), but M-H is never an equilibrium listing order otherwise.

**Part 4.** Finally, consider the BB case. We also consider both listing orders here: H-M and M-H.

When the listing order is H-M, the IC conditions are given as follows:

$$\Pi_{H1} = \left(\frac{K_M}{r_{MvM}}\right) r_H v_H - K_H \geq \Pi_{H2} = \left(\frac{K_M}{r_{MvM}}\right) (1-\delta) + \left(1 - \frac{K_M}{r_{MvM}}\right) (r_H v_H - C_1)$$  \hfill (A53)

$$\Pi_{M2} = \left(\frac{K_M}{r_{MvM}}\right) (1-\delta) + \left(1 - \frac{K_M}{r_{MvM}}\right) (r_{MvM} - C_1) \geq \Pi_{M1} = \left(\frac{K_M}{r_{MvM}}\right) r_{MvM} - K_M$$  \hfill (A54)

Since \(\frac{\partial \Pi_{M2}}{\partial r_{MvM}} > 0\), the equilibrium bid is determined at the upper bound defined by (A53). Based on this equilibrium bid, the total advertising cost of Firm H, and the profits of both firms are given as follows:

$$C_H = \frac{r_H v_H K_H + (r_H v_H - C_1) \delta K_M}{r_H v_H + K_H - C_1}$$  \hfill (A55)

$$\Pi_H = \frac{r_H v_H K_H (r_H v_H + K_H - C_1)}{r_H v_H K_H + (r_H v_H - C_1) \delta K_M} - K_H$$  \hfill (A56)

$$\Pi_M = \frac{(r_{MvM} - C_1)(r_H v_H ((1-\delta)K_H + \delta K_M) - \delta(K_H^2 - C_1)(K_H - K_M))}{r_H v_H K_H + (r_H v_H - C_1) \delta K_M}$$  \hfill (A57)

Plugging the equilibrium bid into (A54), we obtain the condition for H-M to be an equilibrium listing order as follows:

$$\frac{(r_{MvM} - C_1)\{(1-\delta)K_H + \delta K_M\}r_H v_H - \delta(K_H^2 - C_1)C_1\}}{r_H v_H K_H - \delta(r_H v_H - C_1)K_H} + \frac{(r_H v_H + K_H - C_1)r_{MvM} K_M}{K_M \geq 0} \hfill (A58)$$

In addition, in equilibrium, both \(C_H \leq K_H\) and \(C_H \leq K_M\) should hold and thus based on (A55), we have the following additional conditions:

$$K_H(K_H - C_1) - \delta K_M(r_H v_H - C_1) < 0$$  \hfill (A59)

$$K_H(r_H v_H - K_M) - (1-\delta)K_M(r_H v_H - C_1) > 0.$$  \hfill (A60)

When the listing order is M-H, the IC conditions are given as follows:

$$\Pi_{M1} = \left(\frac{K_M}{r_H v_H}\right) r_{MvM} - K_M \geq \Pi_{M2} = \left(\frac{K_M}{r_H v_H}\right) (1-\delta) + \left(1 - \frac{K_M}{r_H v_H}\right) (r_{MvM} - C_1)$$  \hfill (A61)

$$\Pi_{H2} = \left(\frac{K_M}{r_H v_H}\right) (1-\delta) + \left(1 - \frac{K_M}{r_H v_H}\right) (r_H v_H - C_1) \geq \Pi_{H1} = \left(\frac{K_M}{r_H v_H}\right) r_H v_H - K_H$$  \hfill (A62)
Since $\frac{\partial \Pi_H}{\partial w_H} > 0$, the equilibrium bid is determined at the upper bound defined by (A61). Based on this equilibrium bid, the total advertising cost of Firm H, and the profits of both firms are given as follows:

$$
C_M = \frac{r_M v_M K_M + (r_M v_M - C_1) \delta K_H}{r_M v_M + K_M - C_1} \quad (A63)
$$

$$
\Pi_H = \frac{(r_H v_H - C_1) (r_M v_M ((1 - \delta) K_H + \delta K_H) - \delta (K_H^2 - (K_H - K_M)^2))}{r_M v_M + (r_M v_M - C_1) \delta K_H} \quad (A64)
$$

$$
\Pi_M = \frac{r_M v_M K_M (r_M v_M + K_M - C_1)}{r_M v_M + (r_M v_M - C_1) \delta K_H} - K_M \quad (A65)
$$

Plugging the equilibrium bid into (A62), we obtain the condition for H-M to be an equilibrium listing order as follows:

$$
\left\{ \frac{(r_M v_M - C_1)}{r_H v_H + \delta (K_H - K_M) C_1} \right\} + (r_H v_H + K_H - C_1) r_M v_M K_M + K_M \leq 0 \quad (A66)
$$

In addition, in equilibrium, both $C_M \leq K_H$ and $C_M \leq K_M$ should hold and thus based on (A63), we have the following additional conditions:

$$
K_M (K_M - C_1) - \delta K_H (r_M v_M - C_1) < 0 \quad (A67)
$$

$$
K_M (r_M v_M - K_H) - (1 - \delta) K_H (r_M v_M - C_1) > 0. \quad (A68)
$$

\[\square\]

**Proof of Proposition 5.** We solve the game using backward induction. In the second period of the bidding game, depending on the result of the first period, there are two subgames: Subgame 1 after (H-M) in the first period, and Subgame 2 after (M-H) in the first period. We solve these two subgames in Part 1, and then move to the first-period game in Part 2.

**Part 1.** Let $K_i$ denote Firm $i$'s budget at the beginning of the second period. In this part, we solve the two subgames of Period 2. First consider Subgame 1 (after Firm H takes the first slot and Firm M takes the second slot in the first period). Note that since we consider the case with $\delta r_H v_H + 2(1 - \delta) C_1 \leq K < 2\delta r_H v_H + 2(1 - \delta) C_1$, Firm H's remaining budget is constrained at the first slot but not at the second slot, while Firm M's residual budget is not constrained at either slot. Then by Part 2 of Lemma A1, we have the following equilibrium results: when $K_H \geq K_H^{(1)}$, Firm H takes the first slot in Period 2 and, the equilibrium cost and profits are given by,

$$
C_{H2}^{(1)} = \frac{K_H r_H v_H}{K_H + (1 - \delta) (r_H v_H - C_1)} \quad (A69)
$$

$$
\Pi_H^{(1)} = (1 - \delta) (r_H v_H - C_1) \quad (A70)
$$

$$
\Pi_M^{(1)} = \frac{r_M v_M - C_1}{r_H v_H} \left\{ r_H v_H - \delta K_H - \delta (1 - \delta) \left( r_H v_H - C_1 \right) \right\} \quad (A71)
$$
where \( C_t \) refers to Firm \( i \)'s advertising cost in Period \( t \). When \( K_H < K_H^{(1)} \), Firm H takes the second slot in Period 2 and, the equilibrium cost and profits are given by,

\[
C_M^{(2)} = \frac{K_H r_H v_H}{K_H + (1-\delta)(r_H v_H - C_1)} \tag{A72}
\]

\[
\Pi_H^{(2)} = (1 - \delta)(r_H v_H - C_1) \tag{A73}
\]

\[
\Pi_M^{(2)} = r_M v_M - \frac{K_H r_H v_H}{K_H + (1-\delta)(r_H v_H - C_1)}. \tag{A74}
\]

Note that across the two cases, Firm H’s profits remain the same due to Firm M’s cost-raising bid at the second slot.

Next we consider Subgame 2 (after Firm H takes the second slot and Firm M takes the first slot in the first period). In this case, Firm M’s second-period budget constraint is binding at the first slot (but not at the second slot) while Firm H faces no budget constraint at any slot. Then by Part 3 of Lemma A1, we have the following results: since we consider the case where \( r_M v_M - C_1 \leq 4\delta(1-\delta)(r_H v_H - C_1) \) holds, Firm H always takes the first slot in the second period and the equilibrium cost and profits are given by,

\[
C_H^{(3)} = \frac{K_{HM} v_M v_M}{K_M + (1-\delta)(r_M v_M - C_1)} \tag{A75}
\]

\[
\Pi_H^{(3)} = r_H v_H - \frac{K_{HM} r_M v_M}{K_M + (1-\delta)(r_M v_M - C_1)} \tag{A76}
\]

\[
\Pi_M^{(3)} = (1 - \delta)(r_M v_M - C_1). \tag{A77}
\]

**Part 2.** In this part, we solve the first-period game given the results of the second-period subgames. First suppose Firm H takes the first slot in the equilibrium of the first period. Then, it earns \( r_H v_H - C_{H1} \) in the first period, and \( (1 - \delta)(r_H v_H - C_1) \) in the second period regardless of the slot it takes in the second period. In this case, since Firm M takes the second slot in the first period, its profits are \( (1 - \delta)(r_M v_M - C_1) \), but in the second period, depending on the slot it takes, it earns \( \Pi_M^{(1)} \) or \( \Pi_M^{(2)} \). We denote this second-period profit by \( \Pi_{M2}(C_{H1}) \) given that Firm H’s residual budget in Period 2 is given as \( K_H = K - C_{H1} \).

If Firm H deviates to the second slot in Period 1, its profits are \( (1 - \delta)(r_H v_H - C_1) \) in the first period. In the second period, according to the above result, Firm H always takes the first slot and earns \( r_H v_H - \frac{K_{HM} v_M v_M}{K + (1-\delta)(r_M v_M - C_1)} \) which we denote by \( \Pi_{H2}(C_{H1}) \) given that Firm M’s residual budget in Period 2 is given as \( K_M = K - C_{H1} \). In this deviation, Firm M also switches to the first slot and thus its profits are given as \( r_M v_M - C_{H1} \) in the first period and \( (1 - \delta)(r_M v_M - C_1) \) in the second period.

Given the discussion so far, the incentive compatibility conditions when the listing order is H-M are given by (with the discount factor being 1),

\[
r_H v_H - C_{H1} + (1 - \delta)(r_H v_H - C_1) \geq (1 - \delta)(r_H v_H - C_1) + \Pi_{H2}(C_{H1}) \tag{A78}
\]

\[
(1 - \delta)(r_M v_M - C_1) + \Pi_{M2}(C_{H1}) \geq r_M v_M - C_{H1} + (1 - \delta)(r_M v_M - C_1). \tag{A79}
\]
where $\Pi_{M2}(C_{H1})$ is Firm M’s second period equilibrium profit and $\Pi_{H2}(C_{H1})$ is Firm H’s second period deviation profit, and they are given as

$$
\begin{align*}
\Pi_{M2}(C_{H1}) & = \begin{cases} 
\Pi_1^{(1)} = \frac{r_M v_M}{H} + (1-\delta)\frac{r_H v_H C_{M}(1)}{K_H} & \text{when } K - C_{H1} \geq K_{H}^{(1)} \\
\Pi_2^{(1)} = r_M v_M - \frac{K_H v_H C_{M}(1)}{K_M + (1-\delta)\frac{r_H v_H C_{M}(1)}{K_H}} & \text{when } K - C_{H1} < K_{H}^{(1)}
\end{cases} \\
\Pi_{H2}(C_{H1}) & = \begin{cases} 
\Pi_3^{(1)} = r_H v_H - \frac{K_M r_M v_M}{K_M + (1-\delta)\frac{r_H v_H C_{M}(1)}{K_H}} & \text{when } K - C_{H1} \geq K_{H}^{(1)}
\end{cases}
\end{align*}
$$

These conditions can be further reduced to,

$$
\begin{align*}
& r_H v_H - C_{H1} \geq \Pi_{H2}(C_{H1}) \tag{A82} \\
& \Pi_{M2}(C_{H1}) \geq r_M v_M - C_{H1}. \tag{A83}
\end{align*}
$$

Now observe that in (A82), the left-hand side decreases with $C_{H1}$ while the right-hand side increases with $C_{H1}$ (noting that $K_H = K - C_{H1}$ on the right-hand side) and that in (A83), the left-hand side increases with $C_{H1}$ while the right-hand side decreases with $C_{H1}$ (noting that $K_H = K - C_{H1}$ on the left-hand side). Then, it is easy to see that (A82) defines the upper bound while (A83) defines the lower bound of $C_{H1}$. Now, since $\delta \Pi_{M2}(K_{H}^{(1)}) > 0$, Firm M’s equilibrium bid in the first period is determined at the highest possible level and so does Firm H’s first-period cost, which, by (A82), is given as:

$$
C_{H1}^* = \frac{K + (2-\delta) r_M v_M - (1-\delta) C_1 + \sqrt{(K + (2-\delta) r_M v_M - (1-\delta) C_1)^2 - 4 K_H v_H}}{2} \tag{A84}
$$

Given this solution, $K - C_{H1} \geq K_{H}^{(1)}$ becomes equivalent to $K \geq K^*$.

This implies that after Firm H takes the first slot in the first period, the second-period listing order is given as H-M if $K \geq K^*$ but M-H otherwise.

Moving attention to the first period, the equilibrium listing order is H-M if (A83) holds at the solution given in (A84), i.e., if $\Pi_{M2}(C_{H1}^*) \geq r_M v_M - C_{H1}^*$ holds, but M-H otherwise. Given the second-period result, this condition becomes (1) $\Pi_{M}^{(1)}(C_{H1}^*) \geq r_M v_M - C_{H1}^*$ when $K \geq K^*$ and (2) $\Pi_{M}^{(2)}(C_{H1}) \geq r_M v_M - C_{H1}^*$ otherwise, which are respectively equivalent to (1) $K^{**} < K \leq K^{***}$ and to (2) $K \leq (2-\delta) C_1$, where

$$
\begin{align*}
K^{**} & = \frac{r_H v_H ((1-\delta)(1-\delta) r_H v_H + (1-\delta)(1-\delta) r_M v_M + (1-\delta)(1-\delta) r_M v_M) v_M - \delta^2 (1-\delta)(1-\delta) r_M v_M v_M}{2 \delta r_H v_H} \tag{A85} \\
K^{***} & = \frac{r_H v_H ((1-\delta)(1-\delta) r_H v_H + (1-\delta)(1-\delta) r_M v_M + (1-\delta)(1-\delta) r_M v_M) v_M - \delta^2 (1-\delta)(1-\delta) r_M v_M v_M}{2 \delta^2 r_H v_H} + \frac{1}{2 \delta^2 r_H v_H}
\end{align*}
$$

Thus, when $K \geq K^*$ (i.e., when the second-period listing order is H-M in Subgame 1), the first-period listing order is given as H-M if $K^{**} < K \leq K^{***}$ but as M-H otherwise; when

---

2Although omitted here, we can explicitly derive the condition for M-H to be an equilibrium listing order in the first period, by considering the incentive compatibility conditions associated with the M-H order. It turns out that this condition is identical to the current condition except for the sign.
$K < K^*$ (i.e., when the second-period listing order is M-H in Subgame 1), the first-period listing order is always given as M-H (since the value $K$ in our consideration is confined to $K \geq \delta r_M v_M + 2(1 - \delta)C_1$ which is always greater than $(2 - \delta)C_1$).³

Therefore, Firm H takes the first slot in both periods (and Firm M takes the second slot in both periods) if $\max\{K^*, K^{**}\} < K < K^{***}$. If $K < \max\{K^*, K^{**}\}$ or $K > K^{***}$ holds, Firm H takes the second slot in the first period and then moves up to the first slot in the second period while Firm M shows the opposite pattern.

Finally, depending on the parameter values, each of $K^*$, $K^{**}$, and $K^{***}$ may fall either within the range of consideration (i.e., $[\delta r_M v_M + 2(1 - \delta)C_1, 2\delta r_M v_M + 2(1 - \delta)C_1]$) or out of this range. However, there exists at least one case of all of $K^*$, $K^{**}$, and $K^{***}$ falling in the range. For example, when $r_H = r_M = 0.4, v_H = 10.2, v_M = 10.1, \delta = 0.8$ and $C_1 = 4$, we have $K^* = 4.88$, $K^{**} = 5.01$, and $K^{***} = 8.11$, and the range of consideration is given by $[4.86, 8.13]$. This completes the proof. □

**Proof of Proposition 6**

In this proposition, we derive the optimal budget-setting strategies for both firms. In Lemmas A2 and A3, we first rule out some dominated strategies. We then prove that the remaining strategies constitute a budget-setting equilibrium.

**Lemma A2.** When allowed to set the budget, neither firm chooses its budget to be binding at the second slot.

**Proof.** We prove the claim of the lemma in three parts: when the competitor has no budget constraint (in Part 1); when the competitor has a binding budget constraint only at the first slot (in Part 2); and when the competitor’s budget is binding at both slots (in Part 3). To facilitate exposition, we use the three letters, $N$, $B$, and $C$, to respectively denote the decision of each firm to set a non-binding budget ($N$), a budget binding at the first slot ($B$), and a budget binding at the second slot ($C$). Based on this notation, we denote each scenario by combining the two letters representing both firms’ decisions by the order of Firm H’s and Firm M’s decisions (e.g., $NB$, $BC$, etc.).

**Part 1.** In this part, we examine each firm’s incentive to set the budget to be binding at the second slot, when the competitor has no budget constraint (i.e., in $CN$ and $NC$). Suppose Firm H has no binding budget constraint but Firm M chooses the budget to be constrained at the second slot ($K_M < (1 - \delta)C_1$): $NC$. If the equilibrium listing order is H-M, regardless of the equilibrium bid, Firm M’s profit at the second slot is always given by,

$$\Pi_{M2} = \frac{K_M}{(1-\delta)C_1}(1-\delta)r_M v_M - K_M = \frac{K_M}{C_1}(r_M v_M - C_1).$$  \hfill (A87)

³Note that in the latter case, Subgame 2 is played in Period 2 and thus the second-period equilibrium listing order is H-M by Part 1.
Note that the equilibrium listing order can never be M-H. This is because under this listing order, Firm H would have an incentive to raise its bid and thus the equilibrium bid is determined at \( b_H = \frac{C_H}{r_H} \) by equating \( \Pi_{M1} = \frac{K_M}{r_M b_H} r_M v_M - K_M \) and \( \Pi_{M2} = \frac{K_M}{(1-\delta)C_1} (1-\delta) r_M v_M - K_M \), but at this solution, Firm H will never remain in the second slot, since \( \Pi_{H2} = \{ \frac{K_H}{C_1} (1-\delta) + (1 - \frac{K_M}{C_1}) \} (r_M v_M - C_1) \). Therefore, Firm M’s profit when it chooses to be budget-constrained at the second slot is always given as

\[
\Pi_{M}^{NC} = \frac{K_H}{C_1} (r_M v_M - C_1). \tag{A88}\]

This can be easily shown to be less than \( \Pi_{M}^{NN} = (1-\delta)(r_M v_M - C_1) \) (which was derived in Lemma A1, Part 1), for any \( K_M \) less than \( (1-\delta)C_1 \). This implies that Firm M is better off setting the budget not to be constrained in any slot. Thus, Firm M has no incentive to set its budget to be binding at the second slot when Firm H has no budget constraint.

Next, suppose Firm H chooses to set its budget less than \( (1-\delta)C_1 \), while Firm M’s budget is not constrained in any slot: \( CN \). By the same reasoning as above, the equilibrium listing order cannot be H-M. Then under the listing order of M-H, Firm H’s profit is

\[
\Pi_{H}^{CN} = \frac{K_H}{C_1} (r_H v_H - C_1), \tag{A89}\]

which is less than \( \Pi_{H}^{BN} = (1-\delta)(r_H v_H - C_1) \) (which was derived in Lemma A1, Part 2), for any \( K_H \) less than \( (1-\delta)C_1 \). This implies that Firm H is better off by setting its budget to be constrained only at the first slot. Thus, Firm H also has no incentive to set its budget to be binding at the second slot when Firm M has no binding budget constraint.

Part 2. Now we examine the incentive of each firm to set \( K_i < (1-\delta)C_1 \) when the competitor sets its budget to be constrained only at the first slot (i.e., in \( CB \) and \( BC \)). First suppose Firm M sets \( K_M < (1-\delta)C_1 \) when Firm H sets its budget such that \( (1-\delta)C_1 \leq K_H < \delta r_H v_H + (1-\delta)C_1 \): \( BC \). If the listing order is H-M, regardless of the equilibrium bid, Firm M’s profit when taking the second slot is always given as follows:\(^4\)

\[
\Pi_{M2} = \frac{K_M}{(1-\delta)C_1} (1-\delta) r_M v_M - K_M = \frac{K_M}{C_1} (r_M v_M - C_1). \tag{A90}\]

If the listing order is M-H, due to Firm H’s cost-raising bid, Firm M’s profit at the first slot becomes identical to its profit at the second slot. Thus,

\[
\Pi_{M1} = \Pi_{M2} = \frac{K_M}{C_1} (r_M v_M - C_1). \tag{A91}\]

Therefore, Firm M’s profit when it chooses to be budget-constrained at the second slot is given, regardless of the listing order, by

\[
\Pi_{M}^{BC} = \frac{K_H}{C_1} (r_M v_M - C_1). \tag{A92}\]

\(^4\)Note that even if \( \frac{K_H}{r_M b_M} < \frac{K_M}{(1-\delta)C_1} \), the length of the period after Firm H drops out can be derived from \( \frac{K_H}{r_M b_M} (1-\delta)C_1 + l \cdot C_1 = K_M \), which is equivalent to \( l = \frac{K_M}{C_1} - \frac{(1-\delta)K_H}{r_M b_M} \). Thus, Firm M’s profit is \( \Pi_{M} = \frac{K_H}{r_M b_M} (1-\delta)(r_M v_M - C_1) + (\frac{K_M}{C_1} - \frac{(1-\delta)K_H}{r_M b_M})(r_M v_M - C_1) = \frac{K_H}{C_1} (r_H v_H - C_1) \).
Now, first note that we have $\Pi^{BC}_M < (1 - \delta)(r_Mv_M - C_1)$ for any $K_M$ less than $(1 - \delta)C_1$. Also, from Lemma A1, Firm M’s profit in the BN equilibrium with the listing order H-M (by denoting this case by BN1) is given as,

$$\Pi^{BN1}_M = \frac{(r_Mv_M-C_1)(r_Hv_H-\delta K_H-\delta(1-\delta)(r_Hv_H-C_1))}{r_Hv_H},$$

which can be easily shown to be greater than $(1 - \delta)(r_Mv_M - C_1)$, when $K_H < \delta r_Hv_H + (1 - \delta)C_1$. Thus, we have $\Pi^{BC}_M < \Pi^{BN1}_M$ for any $K_M$ less than $(1 - \delta)C_1$.

Given this, we show that $\Pi^{BC}_M < \Pi^{BN2}_M$ also holds, where BN2 refers to the equilibrium with the listing order M-H. First note from Part 2 of Lemma A1 that $C_H$ in BN1 and $C_M$ in BN2 are identical (see (A31) and (A37)). This suggests that in equilibrium, $r_Mb^{BN1}_M = r_Hb^{BN2}_M$ holds, which in turn implies that $\Pi_{M2}(b^{BN1}_M)$ in (A29) and $\Pi_{M2}(b^{BN2}_H)$ in (A35) are identical. Note that $\Pi_{M2}(b^{BN2}_H)$ in (A29) is Firm M’s equilibrium profit in BN2 ($\Pi^{BN2}_M$). However, Firm M’s equilibrium profit in BN1 ($\Pi^{BN1}_M$) is given as $\Pi_{M1}(b^{BN1}_M)$ in (A35) and by (A35), it is greater than or equal to $\Pi_{M2}(b^{BN2}_H)$ in (A35). Thus, we have $\Pi^{BN2}_M \geq \Pi^{BN1}_M$ whenever the equilibrium is given as BN2, which also implies that $\Pi^{BC}_M \leq \Pi^{BN2}_M$ also holds.

Therefore, regardless of the equilibrium listing order, $\Pi^{BC}_M < \Pi^{BN}_M$ always holds. This implies that Firm M is better off by setting its budget not to be constrained. Thus, Firm M has no incentive to voluntarily constrain its budget at the second slot when Firm H is budget-constrained only at the first slot.

Next we examine the incentive of Firm H to set its budget such that $K_H < (1 - \delta)C_1$ when Firm M sets $K_M$ such that $(1 - \delta)C_1 \leq K_M < \delta r_Mv_M + (1 - \delta)C_1$: $CB$. In this case, by the same arguments as above, the equilibrium profit of Firm H is, regardless of the listing order,

$$\Pi^{CB}_H = \frac{K_H}{c_1} (r_Hv_H - C_1).$$

Since $\Pi^{CB}_H < (1 - \delta)(r_Hv_H - C_1)$ for any $K_H$ less than $(1 - \delta)C_1$, we have $\Pi^{CB}_H < \Pi^{NB2}_H$ where $\Pi^{NB2}_H$ is derived in Part 3 of Lemma A1 as

$$\Pi^{NB2}_H = \frac{(r_Hv_H-C_1)(r_Mv_M-\delta K_M-\delta(1-\delta)(r_Mv_M-C_1))}{r_Mv_M},$$

which can be easily shown to be greater than $(1 - \delta)C_1$. In addition, we can show that $\Pi^{NB1}_H > \Pi^{NB2}_H$ holds when the listing order is given by H-M in NB case, in the same way that we have shown $\Pi^{BN2}_M \geq \Pi^{BN1}_M$ previously. This implies that $\Pi^{CB}_H < \Pi^{NB1}_H$ holds as well, under the condition for the equilibrium to be NB1 (i.e., $K_M \geq K^{(1)}_M$).

Therefore, regardless of the equilibrium listing order, $\Pi^{CB}_H < \Pi^{NB}_H$ always holds. This implies that Firm H is better off by setting its budget not to be constrained. Thus, Firm H will not set its budget to be constrained at the second slot when Firm M’s budget is binding only at the first slot.

Part 3. Finally we show that each firm does not follow the competitor’s strategy when the competitor sets its budget to be binding at the second slot (i.e., $CC$). First suppose Firm
M sets $K_M < (1 - \delta)C_1$ when Firm H sets $K_H < (1 - \delta)C_1$. By the same argument as in Part 2, regardless of the listing order, Firm M’s profit is given by,

$$\Pi^C_M = \frac{K_M}{C_1}(r_M v_M - C_1). \quad \text{(A96)}$$

If Firm M chooses not to be budget-constrained, by Part 1, the equilibrium listing order is always M-H and Firm M’s payment in equilibrium is $r_H b_H = C_1$. Then Firm M’s equilibrium profit is,

$$\Pi^C_M = r_M v_M - \frac{K_H}{(1-\delta)C_1}r_H b_H - \left(1 - \frac{K_M}{(1-\delta)C_1}\right)C_1 = r_M v_M - C_1. \quad \text{(A97)}$$

Since $\Pi^C_M > \Pi^C_M$, for any $K_M$ less than $(1 - \delta)C_1$, Firm M has no incentive to set its budget less than $(1 - \delta)C_1$.

The argument is symmetric for the case of Firm H. In other words, given $K_M < (1 - \delta)C_1$, Firm H does not have any incentive to set its budget lower than $(1 - \delta)C_1$ as well, since $\Pi^N_M = r_H v_H - C_1 > \Pi^C_M = \frac{K_H}{C_1}(r_H v_H - C_1)$ for any $K_H$ less than $(1 - \delta)C_1$.

Therefore across the three parts, regardless of the competitor’s budget, neither firm has any incentive to set the budget to be constrained at the second slot.

**Lemma A3.** When allowed to set the budget, neither firm follows the competitor’s strategy to set the budget to be binding only at the first slot.

**Proof.** We prove the lemma in two parts: in Part 1, we prove for Firm M and in Part 2, we show the case of Firm H. To begin with, denote the listing order H-M by 1 and the listing order M-H by 2.

**Part 1.** Suppose Firm M chooses its budget such that $(1 - \delta)C_1 \leq K_M < \delta r_M v_M + (1 - \delta)C_1$ when Firm H’s budget also satisfies $(1 - \delta)C_1 \leq K_H < \delta r_M v_M + (1 - \delta)C_1$. We prove the proposition by showing that $\Pi^B_M < \Pi^B_M$ regardless of the equilibrium listing order.

First to compare $\Pi^B_M$ and $\Pi^B_M$, we note that in both BB1 and BN1, Firm M raises its bid as high as possible and thus the equilibrium bids are obtained respectively by equating both sides of (A29) and (A53) respectively: for BB1,

$$\Pi_{H1} = \left(\frac{K_H}{r_M b_M}\right)r_H v_H - K_H = \Pi_{H2} = \left(\frac{K_M}{r_M b_M} + (1 - \delta) + \left(1 - \frac{K_M}{r_M b_M}\right)\right)(r_H v_H - C_1), \quad \text{(A98)}$$

and for BN1,

$$\Pi_{H1} = \left(\frac{K_H}{r_M b_M}\right)r_H v_H - K_H = \Pi_{H2} = (1 - \delta)(r_H v_H - C_1). \quad \text{(A99)}$$

Since $\frac{\partial \Pi_{H1}}{\partial b_M} < 0$ in both BB1 and BN1 and $\Pi_{H2}$ is larger in BB1 than in BN1 for any $b_M$ satisfying $r_M b_M > K_M$, the equilibrium bid is higher in BN1 than in BB1: $b_{M1}^{BN1} > b_{M1}^{BB1}$. Then, since $\Pi_{M2}$ as shown in (A29) and (A54) is an identical function of $b_M$ across the two cases (i.e., BB1 and BN1) and since $\frac{\partial \Pi_{M2}}{\partial b_M} > 0$, the equilibrium profit of Firm M is higher in BN1 than in BB1: $\Pi^B_M < \Pi^B_M$. 


Next by the same argument as in Part 2 of the proof for Lemma A2, we have $\Pi_{M}^{BN2} > \Pi_{M}^{BN1}$ when the equilibrium listing order of the BN case is given by M-H (i.e., when $K_{H} < K_{M}^{(1)}$). In this case, $\Pi_{M}^{BB1} < \Pi_{M}^{BN2}$ holds as well.

Finally, we compare $\Pi_{M}^{BB2}$ with $\Pi_{M}^{BN1}$ and $\Pi_{M}^{BN2}$. First note that based on $\Pi_{M}^{BB2}$ in (A62),
\[
\frac{\partial \Pi_{M}^{BB2}}{\partial K_{M}} = \frac{\delta K_{M}(r_{M}v_{M} + c_{1})^{2}(r_{M}v_{M} - \delta K_{M})}{2(\delta K_{H}(r_{M}v_{M} - C_{1}) + r_{M}v_{M})(c_{1} + \sqrt{c_{1}^{2} - 4\delta K_{H}(r_{M}v_{M} - C_{1})})} > 0
\]
holds and that (A67) defines the upper bound of $K_{M}$. Thus, the optimal profit of Firm M in BB2 is less than or equal to the profit when this upper bound is plugged in, namely,
\[
\Pi_{M}^{BB2} \leq \pi_{M}^{0} = \frac{(r_{M}v_{M} - C_{1})(r_{M}v_{M} - \delta K_{H})(C_{1} + \sqrt{C_{1}^{2} - 4\delta K_{H}(r_{M}v_{M} - C_{1})})}{2\delta K_{H}(r_{M}v_{M} - C_{1}) + r_{M}v_{M}(c_{1} + \sqrt{c_{1}^{2} - 4\delta K_{H}(r_{M}v_{M} - C_{1})})}
\]  \[\text{(A100)}\]

It turns out that $\Pi_{M}^{BN1} > \pi_{M}^{0}$ is equivalent to $K_{H} \geq K_{M}^{(1)}$, which is identical to the condition for BN1 to be an equilibrium. At the same time, $\Pi_{M}^{BN2} \geq \pi_{M}^{0}$ is also equivalent to $K_{H} \leq K_{M}^{(1)}$. Therefore, $\Pi_{M}^{BN1} \geq \Pi_{M}^{BB2}$ holds whenever the BN1 equilibrium is defined, while $\Pi_{M}^{BN2} \geq \Pi_{M}^{BB2}$ holds whenever the BN2 equilibrium is defined.

Therefore, regardless of the equilibrium listing order, $\Pi_{M}^{BB} < \Pi_{M}^{BN}$ always holds. This implies that Firm M is better off by choosing the budget not to be constrained in any slot. Thus, Firm M has no incentive to follow Firm H’s strategy to set the budget constraint to be binding at the first slot.

**Part 2.** Suppose Firm H chooses its budget such that $(1 - \delta)C_{1} \leq K_{H} < \delta r_{M}v_{M} + (1 - \delta)C_{1}$ when Firm M’s budget also satisfies $(1 - \delta)C_{1} \leq K_{M} < \delta r_{M}v_{M} + (1 - \delta)C_{1}$. We prove the proposition by showing that $\Pi_{H}^{BB} < \Pi_{H}^{NB}$ regardless of the equilibrium listing order.

First we compare $\Pi_{H}^{BB2}$ and $\Pi_{H}^{NB2}$, noting that in both BB2 and NB2, Firm H raises its bid as high as possible and thus the equilibrium bids are obtained respectively by equating both sides of (A47) and (A61) respectively: for BB2,
\[
\Pi_{M1} = \left(\frac{K_{M}}{r_{M}v_{M}}\right)r_{M}v_{M} - K_{M} = \Pi_{M2} = \left(\frac{K_{H}}{r_{M}v_{H}}\right)(1 - \delta) + \left(1 - \frac{K_{H}}{r_{M}v_{H}}\right)(r_{M}v_{M} - C_{1})
\]  \[\text{(A101)}\]
and for NB2,
\[
\Pi_{M1} = \left(\frac{K_{M}}{r_{M}v_{H}}\right)r_{M}v_{M} - K_{M} = \Pi_{M2} = (1 - \delta)(r_{M}v_{M} - C_{1})
\]  \[\text{(A102)}\]

Since $\frac{\partial \Pi_{M1}}{\partial b_{H}} < 0$ in both BB2 and NB2 and $\Pi_{M2}$ is larger in BB2 than in NB2 for any $b_{H}$ such that $r_{H}v_{H} > K_{H}$, the equilibrium bid is higher in NB2 than in BB2: $b_{H}^{BB2} > b_{H}^{NB2}$. From (A48) and (A62), it is easy to see that $\Pi_{H2}$ is an identical function of $b_{H}$ across the two cases (i.e., NB2 and BB2) and that $\frac{\partial \Pi_{M2}}{\partial b_{H}} > 0$. Thus, from $b_{H}^{BB2} > b_{H}^{NB2}$, we can see that the equilibrium profit of Firm H is higher in NB2 than in BB2: $\Pi_{H}^{BB2} > \Pi_{H}^{NB2}$.

Next, by the same argument as in Part 2 of the proof for Lemma A2, we have $\Pi_{H}^{NB1} > \Pi_{H}^{NB2}$ when the equilibrium listing order of the NB case is given by H-M (i.e., when $K_{M} \leq K_{M}^{(1)}$). Thus, $\Pi_{H}^{BB2} < \Pi_{H}^{NB1}$ also holds in this case.

Finally, to compare $\Pi_{M}^{BB1}$ with $\Pi_{M}^{NB1}$ and $\Pi_{M}^{NB2}$, we note that based on $\Pi_{H}^{BB1}$ in (A53),
\[
\frac{\partial \Pi_{M}^{BB1}}{\partial K_{H}} = \frac{\delta K_{M}(r_{H}v_{H} - C_{1})^{2}(r_{H}v_{H} - \delta K_{M})}{(r_{H}v_{H} - K_{H} + \delta K_{M}(r_{H}v_{H} - C_{1}))^{2}} > 0
\]
holds and that (A59) defines the upper bound of $K_{H}$.
Then the optimal profit of Firm H in BB1 is less than or equal to the profit when this upper bound is plugged in, namely,

$$\Pi_H^{BB1} \leq \pi_H^0 = \frac{(r_Hv_H - C_1)\{r_Hv_H - \delta K_M\}(C_1 + \sqrt{C_1^2 - 4\delta K_M(r_Hv_H - C_1)})}{2\delta K_M(r_Hv_H - C_1) + r_Hv_H\{C_1 + \sqrt{C_1^2 - 4\delta K_M(r_Hv_H - C_1)}\}}$$

(A103)

Now it is easy to see that $\Pi_H^{NB1} \geq \pi_H^0$ is equivalent to $K_M \leq K_M^{(1)}$, which is identical to the condition for NB1 to be an equilibrium, and that $\Pi_H^{NB2} \geq \pi_H^0$ is equivalent to $K_M \geq K_M^{(1)}$, which is identical to the condition for NB2 to be an equilibrium. Therefore $\Pi_H^{NB1} \geq \Pi_H^{BB1}$ holds whenever the NB1 equilibrium is defined, while $\Pi_H^{NB2} \geq \Pi_H^{BB1}$ holds whenever the NB2 equilibrium is defined.

Therefore regardless of the equilibrium listing order, we always have $\Pi_H^{BB} < \Pi_H^{NB}$, which implies that Firm H is better off by choosing the budget not to be constrained in any slot. Thus Firm H has no incentive to follow Firm M’s strategy to set the budget constraint to be binding at the first slot.

Across the two parts, in equilibrium, neither firm will constrain its budget at the first slot, when the competitor does so. □

Proof of Proposition 6. First note that we can divide the strategy space of each firm as follows: (1) $K_i \geq \delta r_M v_M + (1 - \delta)C_1$ (call this case N), (2) $(1 - \delta)C_1 \leq K_i < \delta r_M v_M + (1 - \delta)C_1$ (call it B), and (3) $K_i < (1 - \delta)C_1$ (call it C). Based on this, we can consider 9 (3 by 3) different scenarios as potential equilibrium scenarios. Among them, Lemma A2 rules out CN, CB, CC while Lemma A3 removes BB. Among the remaining three scenarios (BN, NB, and NN), BN is also ruled out, because from (A26), (A32), and (A38), it is easy to see that $\Pi_H^{NN} = r_Hv_H - \{\delta r_M v_M + (1 - \delta)C_1\} > \Pi_B^{BB} = (1 - \delta)(r_Hv_H - C_1)$. Thus, Firm H never chooses to be budget-constrained when Firm M is not.

Now we show that both NN and NB can constitute an equilibrium. First note from above that $\Pi_H^{NN} > \Pi_H^{BN}$ holds. Also as shown in the proof of Lemma A2, we have $\Pi_H^{BN} > \Pi_H^{CN}$. In addition, Lemma A3 shows $\Pi_H^{NB} > \Pi_H^{BB}$ while Lemma A2 establishes $\Pi_H^{BB} > \Pi_H^{CB}$. Thus Firm H has no incentive to deviate from N if Firm M chooses N or B. Next consider Firm M’s incentive. From (A27), (A44), and (A52), we have $\Pi_M^{NN} = \Pi_M^{NB} = (1 - \delta)(r_M v_M - C_1)$ but from Lemma A2, we have seen $\Pi_M^{NN} > \Pi_M^{NC}$. Thus Firm M has no incentive to deviate from N or B when Firm H chooses N. Therefore, both NN and NB are equilibrium scenarios. This implies that in equilibrium, Firm H chooses $K_H \in [r_M v_M + (1 - \delta)C_1, \infty)$ while Firm M sets $K_M \in [(1 - \delta)C_1, \infty)$. Thus in the budget-setting equilibrium, Firm H always chooses not to be budget-constrained, while Firm M may choose to do so at the first slot. □