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A canonical optimal stopping problem for American options under a double exponential jump-diffusion model

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This paper presents a simple numerical approach to compute accurately the values and optimal exercise boundaries for American options when the underlying process is a double exponential jump-diffusion model that prices jump risk. The present work extends the canonical representation for American options initially developed in the Brownian motion set-up. Here, too, jump-diffusion pricing models can be reduced to a single optimal stopping problem, indexed by one more parameter, and linear spline approximations of the stopping boundary in the canonical scale with only a few knots are supported through numerical evidence. These approximations can then be exploited to solve the integral equation defining the early exercise boundary of an American option efficiently and accurately, thus leading to its efficient and accurate pricing and hedging.

1 INTRODUCTION

The geometric Brownian motion assumption in the Black–Scholes–Merton model has been challenged through a large body of empirical studies. They express attempts to capture asymmetric, leptokurtic features in the underlying asset return distribution. Recent statistical effort in this realm has mostly been focused on models that incorporate the occurrence of rare jumps in a process that would otherwise follow a diffusion (cf, Ait-Sahalia (2002), Carr and Wu (2003) and Eraker *et al* (2003)). The basic precept is that the asset price fluctuates most of the time according to standard supply and demand information. These variations are captured through Brownian motion and are amplified, in the form of jumps, when an occasional large impact event occurs. While one of the earliest papers on option pricing (Merton (1976)) addresses such jumps by using a Poisson process model, more recent work has been moving towards incorporating more general Lévy processes (cf, Carr and Wu (2003), Chan (1999) and Madan *et al* (1998)).

Alternative approaches to extending the Black–Scholes process assumption include stochastic volatility and GARCH specifications, constant elasticity of variance (CEV), affine stochastic volatility and affine jump diffusion models (cf, Carr *et al* (2003) and Hull (2000) for further details.)

A feature shared by all these models is market incompleteness, where there is no longer a unique risk-neutral pricing probability measure as in the Black–Scholes model. This phenomenon has been dealt with in different ways. Merton (1976) simply assumes that the jumps are not priced but diversified away. Other alternatives make use of risk premia for jumps or volatility that are determined empirically.

Another feature shared by all of these models is their limitation to mostly European options, hindering their applicability to path dependent options, such as American, lookback and barrier options.

A recent formulation due to Kou (2002) offers several advantages relative to the above. It is realistic enough to capture asymmetric, leptokurtic features in the underlying assert return distribution, and simple enough to allow tractability, leading to analytical expressions for European, lookback, barrier and perpetual American options (Kou and Wang (2004)). In this model, the logarithm of the asset price follows a Brownian motion plus a compound Poisson process with jump sizes that have a double exponential distribution and the pricing is based on a rational expectations argument.

In this paper, we develop a simple numerical method to price accurately finite-lived American options using the aforementioned model. This method, based on a canonical representation of the pricing problem, yields accurate optimal exercise boundaries and enables us to support linear spline approximations of the stopping boundary in the canonical (Brownian) scale, with only a few knots (typically three or four). An implication of this is the ability to now solve accurately and efficiently the integral equation defining the early exercise boundary of an American option. The present work extends the canonical representation for American options initially developed in the Brownian motion set-up (AitSahlia and Lai (1999/2000)).

Similarly to the original Brownian context, the space–time transformation applied to the present jump-diffusion model provides additional benefits. It reduces a whole class of American option valuation problems to a *single* canonical optimal stopping problem indexed by two parameters in the absence of dividends and by three parameters in the presence of an additional dividend rate. This stopping problem is solved over a time horizon that is significantly smaller than the original as the calendar time is scaled by σ^2, where σ (in a typical practical range of $(0.1, 0.4)$) is the diffusion volatility of the underlying asset return in the absence of jumps.

Section 2 reviews the double-exponential jump-diffusion model of Kou (2002). Section 3 describes the reduction of the American option pricing problem under this model to a canonical optimal stopping problem. Section 4 describes the Bernoulli walk method with jumps for solving such optimal stopping problems. It is an extension of that which was initially developed by AitSahlia and Lai (1999/2000) when the underlying follows a geometric Brownian motion. Through numerical cases, Section 5 illustrates this approach to instances of practical interest. In particular, linear splines, with only a few knots, are shown to be good approximations to the optimal stopping boundary in the canonical scale. Section 6 concludes.

2 THE DOUBLE-EXPONENTIAL JUMP-DIFFUSION MODEL

The price process of the underlying is assumed to solve the following stochastic differential equation (SDE):

$$\frac{\mathrm{d}S(t)}{S(t-)} = (\mu - d)\,\mathrm{d}t + \sigma\,\mathrm{d}W(t) + d\left(\sum_{i=1}^{N(t)}(V_i - 1)\right) \tag{1}$$

with μ, d and σ being the expected return, dividend rate and diffusion volatility, respectively, of the underlying asset, and where W is a standard Brownian motion, N is a Poisson process with rate λ, $(V_i)_i$ is a sequence of independent and identically distributed (iid) positive random variables such that the jump $Y = \log V$ has asymmetric double exponential distribution with density:

$$f_Y(y) = p\eta_1\,\mathrm{e}^{-\eta_1 y}1_{\{y\geq 0\}} + q\eta_2\,\mathrm{e}^{\eta_2 y}1_{\{y<0\}}$$

Here, $p, q \geq 0$, $p + q = 1$, with p and q representing, respectively, the probability of positive and negative jumps. The parameters η_1 and η_2 are assumed to be such that $\eta_1 > 1$, $\eta_2 > 0$, owing to the fact that:

$$E[V] = E[\mathrm{e}^Y] = (1-p)\frac{\eta_2}{\eta_2+1} + p\frac{\eta_1}{\eta_1 - 1}$$

In addition, $W(t)$, $N(t)$ and $\{V_i\}$ are assumed independent. That Y captures jumps can be seen through standard discretization of Equation (1). Over a small interval Δt:

$$\frac{\Delta S(t)}{S(t)} \approx (\mu - d)\Delta t + \sigma Z\sqrt{\Delta t} + BY$$

where $Z \sim N(0, 1)$ and:

$$B = \begin{cases} 1 & \text{with probability } \lambda\Delta t \\ 0 & \text{with probability } 1 - \lambda\Delta t \end{cases}$$

with Y following the double exponential distribution previously defined.

The presence of jumps renders the market incomplete and one may appeal to a rational expectations argument with a hyperbolic absolute risk aversion (HARA)-type utility function for the representative agent (cf, Lucas (1978) and Naik and Lee (1990)). Following this approach, Kou (2002) showed that one can select a risk-neutral measure P^* such that the equilibrium price of the option is given as the expected value of its discounted payoff under P^*. Under this measure, the asset price $S(t)$ still follows a double exponential jump-diffusion process:

$$\frac{\mathrm{d}S(t)}{S(t-)} = (r - d - \lambda^* k^*)\,\mathrm{d}t + \sigma\,\mathrm{d}W^*(t) + d\left(\sum_{i=1}^{N^*(t)}(V_i^* - 1)\right) \tag{2}$$

where, under P^*, W^* is a standard Brownian motion, N^* is a Poisson process with intensity λ^* and $V_i^* = e^{Y_i^*}$, with $(Y_i^*)_i$ being a sequence of iid random variables that follow a double exponential distribution with parameters $(p^*, \eta_1^*, \eta_2^*)$ and $k^* = E[V^*] - 1$. In other words, the density function of Y^* is:

$$f_{Y^*}(y) = p^* \eta_1^* \, e^{-\eta_1^* y} 1_{\{y \geq 0\}} + (1 - p^*) \eta_2^* \, e^{\eta_2^* y} 1_{\{y<0\}}$$

with $0 \leq p^* \leq 1$, $\lambda^* > 0$, $\eta_1^* > 1$, $\eta_2^* > 0$ and:

$$k^* = E[V^*] - 1 = p^* \frac{\eta_1^*}{\eta_1^* - 1} + (1 - p^*) \frac{\eta_2^*}{\eta_2^* + 1} - 1$$

The starred (*) parameters all depend on the utility function of the representative agent and the processes $N^*(t)$, $W^*(t)$ and the Y^* are still independent under the measure P^*. For ease of notation, we drop the superscript * hereafter.

3 A CANONICAL OPTIMAL STOPPING PROBLEM

The price U of an American option expiring at T with strike K solves the optimal stopping problem

$$U(t, S) = \sup_{\tau \in \mathcal{T}_{[t,T]}} E[e^{-r(\tau - t)} f(S_\tau) \mid S_t = S] \tag{3}$$

where the exercise payoff $f(S)$ is $\max(K - S, 0)$ for a put and $\max(S - K, 0)$ for a call, and $\mathcal{T}_{[t,T]}$ is the set of stopping times in the interval $[t, T]$. As shown by Kou (2002), the expectation above is taken under the measure of the representative agent (with the asterisk dropped from P^*).

The domain of the underlying jump-diffusion process can be partitioned into a continuation region $\mathcal{C}$, where it is optimal to let the process evolve, and its complement $\mathcal{E}$, where it is optimal to stop. The optimal stopping boundary is then that which separates $\mathcal{C}$ and $\mathcal{E}$ (cf, for example, Peskir and Shiryaev (2006) and Pham (1997)). Formally, we can write that U satisfies:

$$\begin{aligned} U(t, S) &= f(P) \quad \text{if } (t, S) \in \mathcal{E} \\ U(t, S) &> f(P) \quad \text{if } (t, S) \in \mathcal{C} \end{aligned}$$

In the continuation region $\mathcal{C}$, it can be shown (cf, Merton (1976) and Pham (1997)) that $U(t, S)$ must satisfy the integro-partial differential equation (IPDE):

$$\begin{aligned} &-\frac{\partial U}{\partial t} + rU - (r - d - \lambda k) S \frac{\partial U}{\partial S} \\ &\quad - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} - \lambda E[U(SV, t) - U(S, t)] = 0 \end{aligned}$$

The stopping problem (3) can then be reformulated analytically as the variational inequality problem of finding $\mathcal{E}$ and U such that:

$$(t, S) \in \mathcal{E} \Rightarrow U(t, S) = f(S)$$

$$(t, S) \notin \mathcal{E} \Rightarrow \begin{cases} U(t, S) > f(S) \\ -\dfrac{\partial U}{\partial t} + rU - (r - d - \lambda k)S\dfrac{\partial U}{\partial S} - \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 U}{\partial S^2} \\ -\lambda E[U(SV, t) - U(S, t)] = 0 \end{cases} \tag{4}$$

We are now ready to reduce the number of parameters $(K, T, r, d, \sigma, \lambda)$ by certain space–time transformations. Dividing all prices by K is akin to assuming that $K = 1$ for all problems, which we do assume henceforth. Consider next the following change of variables:

$$\begin{gathered} t' = t - T \\ y = \frac{1}{\sigma}\left(\ln S - (r - d - \lambda k - \frac{1}{2}\sigma^2)t'\right) \\ u(t', y) = U(t(t'), S(t', y)) \end{gathered} \tag{5}$$

As a result, problem (4) is now re-expressed as:

$$(t', y) \in \Sigma \Rightarrow u(t', y) = h(t', y)$$

$$(t', y) \notin \Sigma \Rightarrow \begin{cases} u(t', y) > h(t', y) \\ \dfrac{\partial u}{\partial t'} + \dfrac{1}{2}\dfrac{\partial^2 u}{\partial y^2} + \lambda E\left[u\left(y + \dfrac{1}{\sigma}\ln V, t'\right) - u(y, t')\right] = 0 \end{cases} \tag{6}$$

where Σ represents the stopping set in the new system of coordinates and:

$$h(t', y) = \begin{cases} \mathrm{e}^{rt'} \max(1 - \mathrm{e}^{(r-d-\lambda k-\frac{1}{2}\sigma^2)t'+\sigma y}, 0) & \text{for a put} \\ \mathrm{e}^{rt'} \max(\mathrm{e}^{(r-d-\lambda k-\frac{1}{2}\sigma^2)t'+\sigma y} - 1, 0) & \text{for a call} \end{cases}$$

Note that:

$$\frac{\partial f(t'y)}{\partial t'} + \frac{1}{2}\frac{\partial^2 f(t', y)}{\partial y^2} + \lambda E\left[f\left(t', y + \frac{1}{\sigma}\ln V\right) - f(t', y)\right]$$

is the infinitesimal generator (applied to f) for a jump-diffusion process X defined through the SDE:

$$\mathrm{d}X(t') = \mathrm{d}W(t') + \mathrm{d}q(t')$$

where W is a standard Brownian motion and q a compound Poisson process with intensity λ and jump size $(1/\sigma)\ln V$.

Hence, we have expressed the valuation of the American option as the following optimal stopping problem:

$$u(t', y) = \sup_{\tau \in \mathcal{T}_{[t',0]}} E[h(\tau, X_\tau) \mid X_{t'} = y] \tag{7}$$

which leads to the ultimate optimal stopping formulation of the American option pricing problem as follows:

THEOREM 1 *Define parameters* $\rho = r/\sigma^2$, $\alpha = d/r$ *and* $\gamma = \lambda/\sigma^2$. *Let* $s = \sigma^2(t - T)$ *and* $z = \ln(S/K) - [\rho(1 - \alpha - \gamma k) - \frac{1}{2}]s$, *where:*

$$k = p\frac{\eta_1}{\eta_1 - 1} + (1 - p)\frac{\eta_2}{\eta_2 + 1} - 1$$

with p, η_1 *and* η_2 *being the parameters associated with the double exponential jump* V *in the SDE (2). Then the American option pricing problem (3) under the original jump-diffusion dynamics in SDE (1) can be reformulated as the optimal stopping problem:*

$$w(s, z) = \sup_{\tau \in \mathcal{T}_{(s,0)}} E[g(\tau, X(\tau)) \mid X(s) = z] \tag{8}$$

where:

$$g(s, z) = \begin{cases} e^{\rho s} \max(1 - e^{(\rho - \alpha\rho - \gamma k - \frac{1}{2})s + z}, 0) & \textit{for a put} \\ e^{\rho s} \max(e^{(\rho - \alpha\rho - \gamma k - \frac{1}{2})s + z} - 1, 0) & \textit{for a call} \end{cases}$$

and $X(s)$ *is the jump-diffusion defined by the SDE:*

$$\mathrm{d}X(s) = \mathrm{d}W(s) + \mathrm{d}q(s) \tag{9}$$

with $W(s)$ *a standard Brownian motion and* $q(s)$ *a Poisson-driven jump process with rate* γ *and jump size* $\ln V$ *such that* $W(s)$, $q(s)$ *and* V *are independent.*

PROOF If we define $w(s, z) \equiv w(s(t'), z(y)) = u(t', y)$ we obtain:

$$\begin{aligned} \frac{\partial u}{\partial t'} &= \sigma^2 \frac{\partial w}{\partial s} \\ \frac{\partial^2 u}{\partial y^2} &= \sigma^2 \frac{\partial^2 w}{\partial z^2} \\ u\left(t', y + \frac{1}{\sigma}\ln V\right) &= w(s, z + \ln V) \end{aligned} \tag{10}$$

Now we use $z = \sigma y$ and Equation (10) to modify problem (6). Division by σ^2 yields:

$$\begin{aligned} &(s, z) \in \mathcal{S} \to w(s, z) = g(s, z) \\ &(s, z) \notin \mathcal{S} \to \begin{cases} w(s, z) > g(s, z) \\ \dfrac{\partial w}{\partial s} + \dfrac{1}{2}\dfrac{\partial^2 w}{\partial z^2} + \dfrac{\lambda}{\sigma^2} E[w(z + \ln V, s) - w(z, s)] = 0 \end{cases} \end{aligned} \tag{11}$$

where the stopping region Σ of problem (6) is mapped to $\mathcal{S}$ under the new system of coordinates. To complete the proof, note that the left-hand side of IPDE (11) is the infinitesimal generator for the process X defined by the SDE (9). □

Finally, we retrieve the solution to the original stopping problem (3) by mapping back, ie,

$$\begin{aligned} S(t) &= K\,\mathrm{e}^{z+[\rho(1-\alpha-\gamma k)-\frac{1}{2}]s} \\ U(t, S) &= K\,\mathrm{e}^{\rho s} w(s, z) \end{aligned} \tag{12}$$

with stopping boundary $\overline{S}(t) = K\,\mathrm{e}^{\overline{z}+[\rho(1-\alpha-\gamma k)-\frac{1}{2}]s}$.

4 NUMERICAL APPROXIMATION: BERNOULLI WALK WITH JUMPS

The optimal stopping problem (8) under jump-diffusion dynamics in SDE (9) is not readily solved and a numerical scheme to generate a solution is therefore needed. To do so, we extend the Bernoulli walk construction of AitSahlia and Lai (1999/2000) to incorporate the jump component of SDE (9). To some extent, our construction will be somewhat similar to that of Amin (1993), who considered a geometric Brownian motion perturbed by Poisson-driven, normally distributed jumps. As in Amin (1993), we first construct the approximating grid, then define the corresponding transition probabilities in order to ensure proper weak convergence to the jump-diffusion process. In contrast to Amin (1993), our Bernoulli construction enables us to: (i) solve simultaneously several option pricing problems identified by the canonical parameters ρ, α and γ over a significantly reduced time interval; and (ii) obtain a correction term for the boundary as developed for diffusions (cf, AitSahlia and Lai (1999/2000) and Lai *et al* (2007)).

Given a small $\delta > 0$, we discretize time and space as follows. Let $s_0 = 0$ and $s_j = s_{j-1} - \delta$ for $j \geq 1$. Next, set:

$$\begin{aligned} \mathbf{Z}_\delta &= \{\sqrt{\delta}n : n \text{ is an integer}\} = \{0, \pm\sqrt{\delta}, \pm 2\sqrt{\delta}, \ldots\} \\ \mathbf{S}_\delta &= \{s_n, s_n - \delta, \ldots, -\delta, 0\} \end{aligned}$$

Had the process X defined by SDE (9) not included the jump component q, we would have approximated problem (8), as in AitSahlia and Lai (1999/2000), by the backward recursion:

$$u(s_i, z) = \max\{g(s_i, z), \tfrac{1}{2}[u(s_{i-1}, z + \sqrt{\delta}) + u(s_{i-1}, z - \sqrt{\delta})]\} \tag{13}$$

where $u(s_0, z) = g(0, z)$, $z \in \mathbf{Z}_\delta$ and $s_{i-1} = s_i + \delta$.

To incorporate the jump component q of SDE (9), we first approximate the possibility of jump in an interval of length δ by $\gamma\delta$ and that of no jump by $1 - \gamma\delta$. We recall that $\{q(t)\}$ is a Poisson-driven jump process with intensity γ and jump size with density function $f_Y(y) = p\eta_1\,\mathrm{e}^{-\eta_1 y}1_{\{y\geq 0\}} + (1-p)\eta_2\,\mathrm{e}^{\eta_2 y}1_{\{y<0\}}$. Let X_n^δ be the position of the approximating discrete process after the nth transition. On the grid $\mathbf{Z}_\delta$, a jump occurs at the $(n+1)$th transition if $X_{n+1}^\delta - X_n^\delta = \pm l\sqrt{\delta}$, where $l \geq 2$. As in Amin (1993), we approximate the jump distribution on the entire real line by partitioning it into intervals of length $\sqrt{\delta}$, the probability measure of which is set to the probability mass assigned to their corresponding midpoints. Specifically, with F defined as the cumulative probability distribution function of Y, we set:

$$\mathrm{d}F(l) = F((l + \tfrac{1}{2})\sqrt{\delta}) - F((l - \tfrac{1}{2})\sqrt{\delta})$$

for $|l| > 1$. More explicitly:

$$\mathrm{d}F(l) = \begin{cases} (1-p)(\mathrm{e}^{\eta_2(l+0.5)\sqrt{\delta}} - \mathrm{e}^{\eta_2(l-0.5)\sqrt{\delta}}) & \text{if } l \le -2 \\ p(1-p)(\mathrm{e}^{-\eta_1(l-0.5)\sqrt{\delta}} - \mathrm{e}^{-\eta_1(l+0.5)\sqrt{\delta}}) & \text{if } l \ge 2 \end{cases}$$

and assign:

$$\mathrm{d}F(0) = F((1 + \tfrac{1}{2})\sqrt{\delta}) - F(-(1 + \tfrac{1}{2})\sqrt{\delta})$$

With Bernoulli increments approximating the diffusion part, we define the following probabilities:

$$P\{X^{\delta}_{n+1} - X^{\delta}_{n} = \pm l\sqrt{\delta}\} = \begin{cases} \frac{1}{2}(1-\gamma\delta) & \text{if } l = 1 \\ \gamma\delta\,\mathrm{d}F(l) & \text{if } l \ne 1 \end{cases}$$

where $\mathrm{d}F(l)$ is defined above for $l \ne 1$. Note that, in practice, we need to truncate the set of possible integer values for l to a finite subset $\{l_{\min}, l_{\min} + 1, \dots, -1, 0, 1, \dots, l_{\max}\}$, where we re-set $\mathrm{d}F(l_{\min})$ and $\mathrm{d}F(l_{\max})$ as:

$$\mathrm{d}F(l_{\min}) = P\{Y < (l_{\min} + 0.5)\sqrt{\delta}\}$$
$$\mathrm{d}F(l_{\max}) = P\{Y > (l_{\max} - 0.5)\sqrt{\delta}\}$$

In turn, the dynamic programming algorithm equivalent of Equation (13) is:

$$u(s_i, z) = \max\Bigg\{g(s_i, z), \tfrac{1}{2}(1-\gamma\delta)[u(s_{i-1}, z + \sqrt{\delta}) + u(s_{i-1}, z - \sqrt{\delta})] + \gamma\delta \sum_{l=l_{\min}, l \ne \pm 1}^{l_{\max}} \mathrm{d}F(l) w(s_{i-1}, z + l\sqrt{\delta})\Bigg\} \tag{14}$$

where $u(s_0, z) = g(0, z)$, $s_{i-1} = s_i + \delta$ and $z \in \tilde{\mathbf{Z}}_\delta$, with $\tilde{\mathbf{Z}}_\delta = \{l_{\min}, \dots, -1, 0, 1, \dots, l_{\max}\}$ being the truncated subset of $\mathbf{Z}_\delta$.

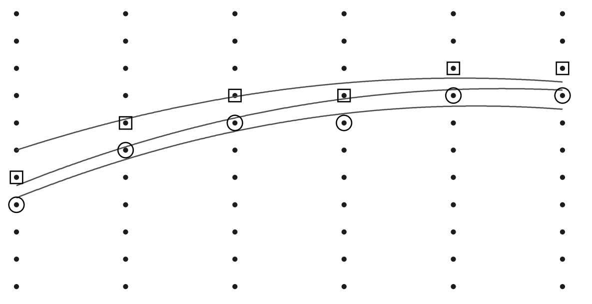
4.1 Corrected boundary approximation

The recursive algorithm (14) classifies grid points on $\mathbf{Z}_\delta$ as either continuation or stopping points. The optimal discrete stopping boundary is then defined, at each date s_i, as:

$$\bar{z}_\delta(s_i) = \begin{cases} \max\{z \in \tilde{\mathbf{Z}}_\delta : w(s_i, z) = g(s_i, z)\}, & \text{for a put} \\ \min\{z \in \tilde{\mathbf{Z}}_\delta : w(s_i, z) = g(s_i, z)\}, & \text{for a call} \end{cases}$$

As illustrated in Figure 1, the actual stopping boundary $\bar{z}(s)$ for the continuous-time model may lie in the vicinity of $\bar{z}_\delta(s)$ in a variety of ways. When the

FIGURE 1 Grid point classification by means of discrete dynamic programming. The highest stopping points and lowest continuation points are identified by circles and squares, respectively. The "true" boundary could be any of the lines shown in the figure.



underlying process follows a Brownian motion (no jumps), Chernoff and Petkau (1976) propose a continuity correction approach to obtain an estimate of $\overline{z}(s)$. Their method has now been more explicitly justified and generalized to generic random walk approximations of Brownian motion in a recent paper of Lai *et al* (2007). In the present paper we show that this method can be extended to the jump-diffusion context. To obtain the correction factor, we first define:

$$\overline{z}_{\delta}^{0}(s_i) = \begin{cases} \overline{z}_{\delta}(s_i) + \sqrt{\delta}, & \text{for a put} \\ \overline{z}_{\delta}(s_i) - \sqrt{\delta}, & \text{for a call} \end{cases}$$

and:

$$\overline{z}_{\delta}^{1}(s_i) = \begin{cases} \overline{z}_{\delta}(s_i) + 2\sqrt{\delta}, & \text{for a put} \\ \overline{z}_{\delta}(s_i) - 2\sqrt{\delta}, & \text{for a call} \end{cases}$$

Roughly speaking, $\overline{z}_{\delta}^{0}(s_i)$ and $\overline{z}_{\delta}^{1}(s_i)$ are the continuation points in $\mathbf{Z}_{\delta}$ that are closest and second closest, respectively, to the stopping region at time s_i. The estimate of $\overline{z}(s_i)$ is then obtained as the result of an extrapolation scheme involving $\overline{z}_{\delta}^{0}(s_i)$ and $\overline{z}_{\delta}^{1}(s_i)$ and their corresponding values of:

$$D_j(s_i) := g(s_i, \overline{z}_{\delta}^{j}(s_i)) - w(s_i, \overline{z}_{\delta}^{j}(s_i)), \quad j = 0, 1$$

Specifically:

$$\overline{z}(s_i) = \overline{z}_{\delta}^{0}(s_i) \pm \sqrt{\delta}|D_1(s_i)/(2D_1(s_i) - 4D_0(s_i))| \tag{15}$$

where the $+$ and $-$ signs apply to the call and put, respectively. When the underlying process is a Brownian motion that is approximated by random walk for the discrete dynamic programming algorithm, Lai *et al* (2007) show that the estimate (15) is within $o(\sqrt{\delta})$ of the actual value of $\overline{z}(s_i)$. In the jump-diffusion context of the present paper, the numerical results of the next section suggest that this correction effect is likely to be identical.

TABLE 1 Convergence of stopping boundary and value function in both scales for American put without dividend ($\alpha = 0$) when $s = -0.05$ and $\rho = 0.5$. Double exponential jump parameters are $p = 0.6$, $\eta_1 = 25$ and $\eta_2 = 25$ and the midpoint of the grid is $z = -0.1$. For this point the value of the canonical option, w, is also displayed as well as the actual prices when $K = 100$ and $\sigma = 0.2$.

		δ			
γ		10^{-3}	10^{-4}	10^{-5}	10^{-6}
0	$\bar{z}$	−0.32126	−0.32107	−0.32089	−0.32091
	$\bar{S}$	72.52368	72.53739	72.55016	72.54856
	$w(-0.05, -0.1)$	0.13178	0.13144	0.13146	0.13146
	$U(t, S)$	12.85300	12.81970	12.82180	12.82160
6.25	$\bar{z}$	−0.32772	−0.32738	−0.32778	−0.32781
	$\bar{S}$	72.27332	72.29760	72.26869	72.26703
	$w(-0.05, -0.1)$	0.13061	0.13045	0.13047	0.13047
	$U(t, S)$	12.73810	12.72250	12.72520	12.72510
25	$\bar{z}$	−0.34489	−0.34783	−0.34834	−0.34835
	$\bar{S}$	71.68621	71.47605	71.43918	71.43861
	$w(-0.05, -0.1)$	0.12724	0.12756	0.12761	0.12761
	$U(t, S)$	12.41000	12.44060	12.44580	12.44620
100	$\bar{z}$	−0.42201	−0.42767	−0.42840	−0.42855
	$\bar{S}$	68.80236	68.41418	68.36398	68.35414
	$w(-0.05, -0.1)$	0.11539	0.11728	0.11747	0.11749
	$U(t, S)$	11.25430	11.43860	11.45730	11.45930
300	$\bar{z}$	−0.60939	−0.62889	−0.62974	−0.62991
	$\bar{S}$	62.80338	61.59058	61.53831	61.52798
	$w(-0.05, -0.1)$	0.09126	0.09648	0.09698	0.09704
	$U(t, S)$	8.90077	9.40996	9.45841	9.46414
1,200	$\bar{z}$	−1.66081	−1.41958	−1.41447	−1.41505
	$\bar{S}$	33.82781	43.05650	43.27708	43.25199
	$w(-0.05, -0.1)$	0.03769	0.04771	0.04865	0.04877
	$U(t, S)$	3.67589	4.65347	4.74481	4.75622

5 NUMERICAL ILLUSTRATION: PUTS WITH NO DIVIDEND

Even if the transformation to the canonical problem reduces the number of main parameters, there are still a few left. In particular, the double exponential distribution requires three parameters, p, η_1 and η_2. However, one of the main points of the present paper, namely canonical reduction to an optimal stopping problem with significantly reduced time horizon, is not affected by the choice of the jump distribution. In addition, our numerical experiments have shown that the second main point of the paper, that of linear spline approximation of the boundary by only a few knots, is not affected by the parameter values of the double exponential distribution. We have therefore chosen to use the values of $p = 0.6$, $\eta_1 = 25$ and $\eta_2 = 25$ to allow for easy comparison with other studies.

FIGURE 2 Optimal stopping boundaries ($\delta = 10^{-4}$) for American puts (no dividend) and their piecewise linear approximations in canonical scale for $\gamma = 0$. From bottom to top, plots are for $\rho = 0.01$, 0.1, 0.5 and 2, respectively. Double exponential parameters are $p = 0.6$, $\eta_1 = 25$ and $\eta_2 = 25$.

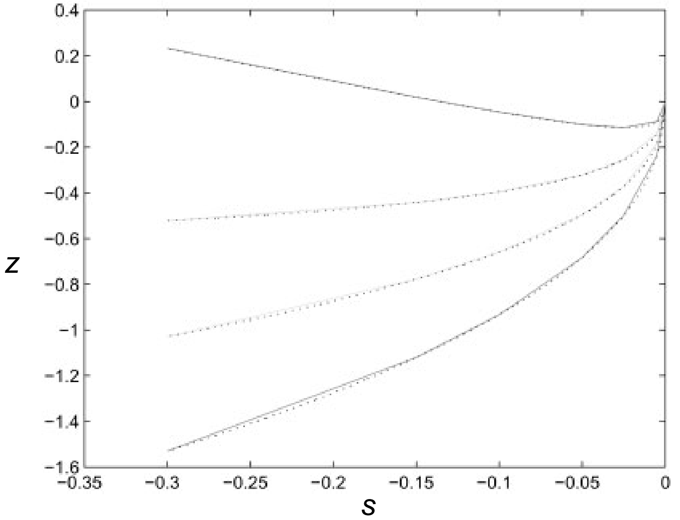


FIGURE 3 Optimal stopping boundaries ($\delta = 10^{-4}$) for American puts (no dividend) and their piecewise linear approximations in canonical scale for $\gamma = 6.25$. From bottom to top, plots are for $\rho = 0.01$, 0.1, 0.5 and 2, respectively. Double exponential parameters are $p = 0.6$, $\eta_1 = 25$ and $\eta_2 = 25$.

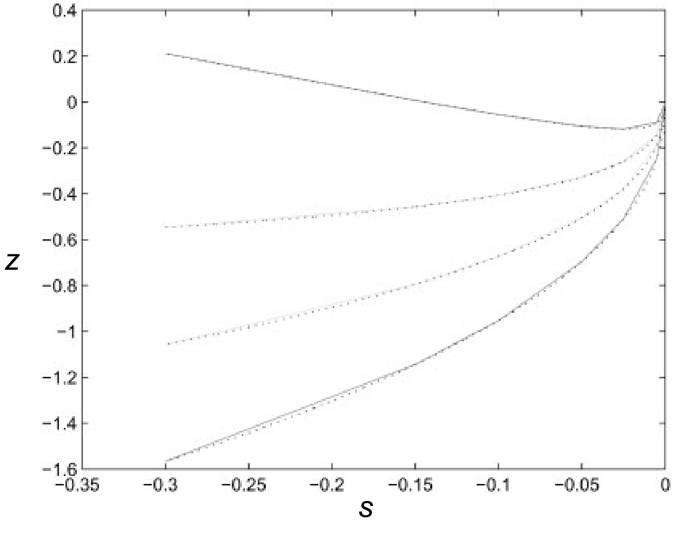


FIGURE 4 Optimal stopping boundaries ($\delta = 10^{-4}$) for American puts (no dividend) and their piecewise linear approximations in canonical scale for $\gamma = 25$. From bottom to top, plots are for $\rho = 0.01$, 0.1, 0.5 and 2, respectively. Double exponential parameters are $p = 0.6$, $\eta_1 = 25$ and $\eta_2 = 25$.

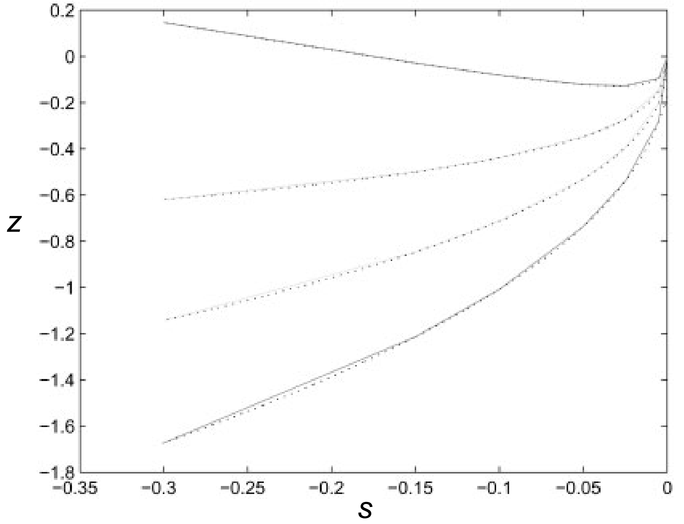


FIGURE 5 Optimal stopping boundaries ($\delta = 10^{-4}$) for American puts (no dividend) and their piecewise linear approximations in canonical scale for $\gamma = 100$. From bottom to top, plots are for $\rho = 0.01$, 0.1, 0.5 and 2, respectively. Double exponential parameters are $p = 0.6$, $\eta_1 = 25$ and $\eta_2 = 25$.

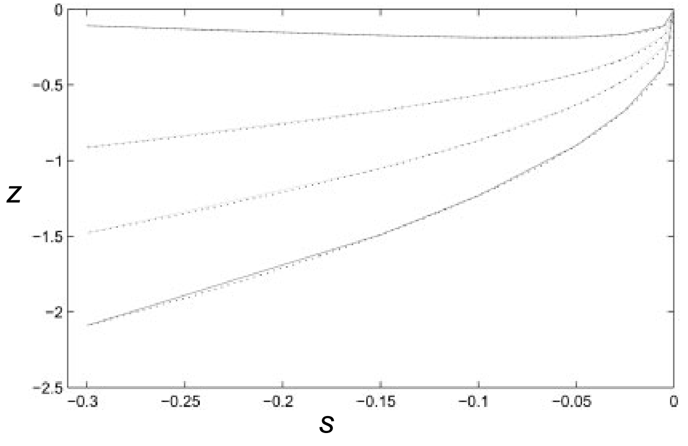


FIGURE 6 Optimal stopping boundaries ($\delta = 10^{-4}$) for American puts (no dividend) and their piecewise linear approximations in canonical scale for $\gamma = 300$. From bottom to top, plots are for $\rho = 0.01$, 0.1, 0.5 and 2, respectively. Double exponential parameters are $p = 0.6$, $\eta_1 = 25$ and $\eta_2 = 25$.

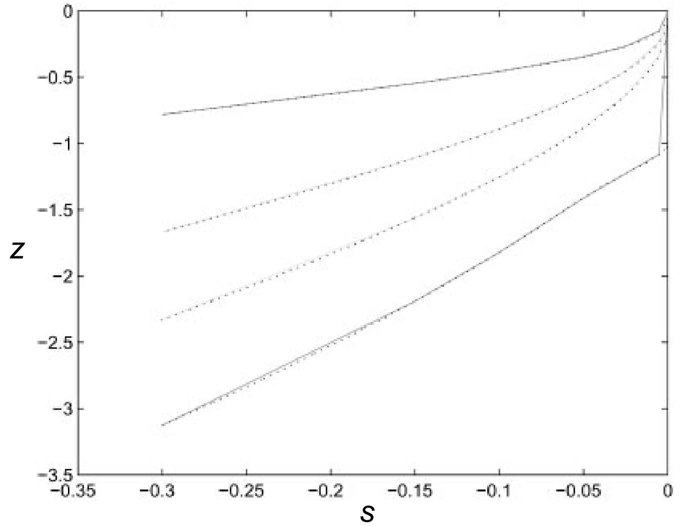


FIGURE 7 Optimal stopping boundaries ($\delta = 10^{-4}$) for American puts (no dividend) and their piecewise linear approximations in canonical scale for $\gamma = 1{,}200$. From bottom to top, plots are for $\rho = 0.01$, 0.1, 0.5 and 2, respectively. Double exponential parameters are $p = 0.6$, $\eta_1 = 25$ and $\eta_2 = 25$.

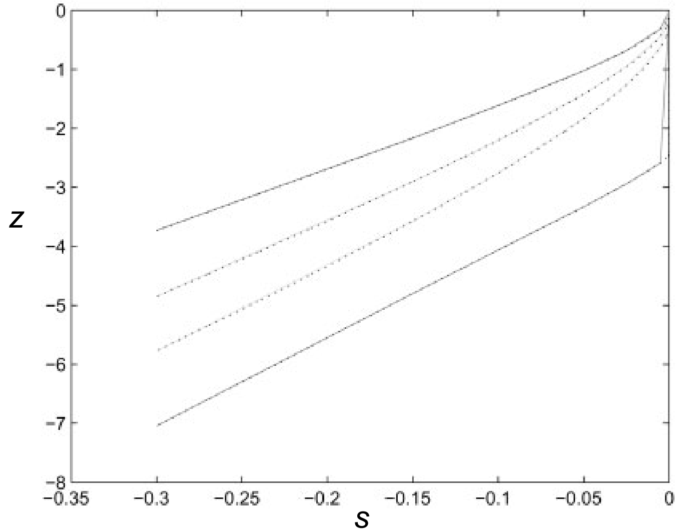


TABLE 2 Boundary knots $\overline{z}(s)$ for linear spline approximations ($\delta = 10^{-4}$), $p = 0.6$, $\eta_1 = 25$, $\eta_2 = 25$ and $\alpha = 0$).

		s					
γ	ρ	**−0.3**	**−0.15**	**−0.1**	**−0.5**	**−0.0.025**	**−0.005**
0	0.01	−1.530370	−1.120210	−0.933143	−0.682444	−0.497858	−0.235654
	0.1	−1.028780	−0.777354	−0.659362	−0.495799	−0.372040	−0.184553
	0.5	−0.522047	−0.442342	−0.395711	−0.321233	−0.254758	−0.140974
	2	0.232720	0.018507	−0.046504	−0.100726	−0.114046	−0.088078
6.25	0.01	−1.566190	−1.143490	−0.952322	−0.695633	−0.507551	−0.245019
	0.1	−1.057480	−0.794782	−0.672786	−0.504443	−0.377438	−0.187513
	0.5	−0.546514	−0.456685	−0.406466	−0.327453	−0.259469	−0.142612
	2	0.211405	0.006589	−0.055323	−0.105842	−0.117367	−0.089388
25	0.01	−1.673290	−1.213770	−1.008160	−0.736274	−0.540292	−0.279217
	0.1	−1.143060	−0.846916	−0.712717	−0.531084	−0.395748	−0.200738
	0.5	−0.620649	−0.500248	−0.439045	−0.347945	−0.272871	−0.147476
	2	0.147231	−0.029071	−0.081678	−0.121926	−0.127085	−0.094205
100	0.01	−2.089730	−1.489320	−1.231480	−0.900004	−0.669388	−0.380812
	0.1	−1.477840	−1.050710	−0.867683	−0.632933	−0.466471	−0.244800
	0.5	−0.912609	−0.671198	−0.566108	−0.427812	−0.324954	−0.173792
	2	−0.107428	−0.172346	−0.185292	−0.184068	−0.166360	−0.109145
300	0.01	−3.126690	−2.191330	−1.822370	−1.412900	−1.227530	−1.082280
	0.1	−2.328380	−1.561690	−1.254340	−0.882850	−0.636498	−0.333343
	0.5	−1.669130	−1.108560	−0.891130	−0.629396	−0.455931	−0.234943
	2	−0.782203	−0.547045	−0.457050	−0.346516	−0.268225	−0.150983
1,200	0.01	−7.044390	−4.799860	−4.062430	−3.330980	−2.941750	−2.584640
	0.1	−5.774770	−3.572990	−2.756330	−1.831580	−1.264860	−0.624638
	0.5	−4.849910	−2.903920	−2.202370	−1.420910	−0.954223	−0.437057
	2	−3.731210	−2.162970	−1.612250	−1.024900	−0.683847	−0.309649

We recall that λ is the annual rate of occurrence of (rare) jumps, with values typically between one per month and one per year ($1 \leq \lambda \leq 12$). With common values of σ between 0.1 and 0.4, the range of values for $\gamma = \lambda/\sigma^2$ is therefore typically from 6.25 to 1,200. Table 1, which also includes $\gamma = 0$ for the no jump situation, illustrates the case for an American put when the underlying asset pays no dividend and where $\rho = 0.5$. It shows that the correction by Equation (15) does contribute to the rapid convergence of the boundary and the value function. In fact, $\delta = 10^{-4}$ already gives accurate results, with the exception of the very frequent jumps case of $\gamma = 1{,}200$, which may be due to the truncation effect of the jump distribution support. Furthermore, Figures 2–7 clearly show that the optimal stopping boundary in the canonical scale is well approximated by a linear spline with only a few knots, six of which are explicitly listed in Table 2. In fact, for typical values of σ between 0.1 and 0.4 and of T between 1/12 and 1/2, the largest value of $|s|$ is $(0.4)^1/2 = 0.08$, and thus at most only four knots will typically be needed.

6 CONCLUSIONS AND DIRECTIONS FOR FURTHER RESEARCH

This paper shows how the canonical optimal stopping problem formulation for American options can be extended from the traditional geometric Brownian motion set-up to include jumps that account for a number of important empirical features such as asymmetric leptokurtic asset returns. As a result, we are able to determine that in the canonical scale, the optimal stopping boundary is well approximated by linear spline with only a few knots. As previously shown by AitSahlia and Lai (2001) in the pure Brownian case, this piecewise linear approximation can be fruitfully exploited for the efficient pricing and hedging of an American option through the use of the pricing integral decomposition formula. When the underlying asset follows a jump-diffusion process, Pham (1997) has derived such a formula and the application of the linear spline approximation to the latter is currently undertaken by one of the authors.

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