This paper studies consumer and market behavior when consumers use price as a cue to quality. We identify situations when this action is appropriate and when it is not appropriate.

I. Introduction

Consumers should use a product's price to determine if the product is affordable. However, consumers also appear to use a product's price as a measure of the product's quality. Many empirical studies (e.g., see reviews by Monroe 1973 and Olson 1977) have shown that when consumers have some uncertainty concerning a product's quality, the consumer often assumes that a higher product price indicates a higher level of quality. Some authors (Gabor and Granger 1966; Tull, Boring, and Gonsior 1964) suggest these studies imply that the traditional economic treatment of price and consumer behavior are wrong.

In studying consumer behavior, we might wonder why consumers consistently use price as a surrogate measure of quality. One explanation might be the objective reality of the price-quality relationship. However, many authors (Friedmand 1967, Sproles 1977, Riesz 1979) have found few positive relationships between product quality ratings given by consumer union publications and the actual brand prices. In fact, negative correlations were often found.

It would appear that there is no objective motivation underlying the prevalent consumer perception of a positive price-quality relationship. This situation is made even more perplexing by other recent findings. Allison-Uhl (1964) found that most consumers could not discern the taste differences among various beer brands yet when Schlitz (Business Week, 1982) lowered the quality of their beer, Schlitz sales dramatically declined. Perhaps, consumer behavior and the price-quality relationship are more complex in a real market than laboratory research suggests.

Several studies reinforce this conclusion. Monroe (1977) suggests the importance of the context in which the relationship occurs. The market, of course, creates the context for the relationship. Geistfeld (1982) notes that real quality and price in a market are both complex concepts because the evaluation of quality varies from publication to publication and the price of a brand varies from outlet to outlet. Finally, Wilde and Schwartz (1979) show that a few consumers can have a dramatic effect on prices in the market.

We conclude that in order to understand consumer perceptions of the price-quality relationship, we must fully understand the environment in which these perceptions are formed. Just as biologists must investigate the environment of the leaf butterfly in order to learn that its appearance is a morphological adaptation to Asian foliage, we must investigate the market environment to better understand consumer behavior in the market.

In a real market, both company behavior and consumer behavior affect product prices. Moreover, consumers exhibit different behaviors and consumers have different impacts on market prices. In this paper, we begin to examine these issues. We will try to identify how consumer behavior interacts with market behavior in order to better understand consumer behavior. Section II begins by studying a market consisting of three types of consumers. We ask whether or not the consumers, who use price as a cue to quality, destroy the price-quality relationship. Section III examines competition and whether or not competition destroys the price-quality relationship. Section IV discusses how consumers may also be using nonprice cues to quality. Section V extends our previous analysis to a multidimensional concept of quality. Finally, section VI provides a summary of our conclusions.

II. The Single Quality Product

In this section, we study a very simple market which will be made more complex in latter sections. We investigate a market with a product which possesses but one quality. The quality might be any aspect of the product as long as the presence of the quality provides some benefit to some, but not necessarily all, consumers.

A. Type 1 Consumers

Some of the consumers in this market are not concerned with this quality of the product. These consumers are very price sensitive. They will buy the lowest priced product regardless of the level of the quality. These consumers are called type 1 consumers. We describe the behavior of the price sensitive, type 1 consumer with equation (1).

\[ q_1 = a - bp \]

where [math]q_1[/math] = the quantity of the product purchased by a type 1 consumer
[math]a[/math] = a positive constant
[math]b[/math] = a positive constant
[math]p[/math] = the price of the product

Type 1 consumers will buy more of the product as the price of the product decreases. The quantity of the product that type 1 consumers buy, is not influenced by the level of the quality of the product.

B. Type 2 Consumers

In order to make our analysis more general, we will allow consumers to exist who do not behave according to the traditional economic model. These consumers are called type 2 consumers and their behavior is described by the equation (2).

\[ q_2 = cQ \]

where [math]q_2[/math] = the quantity of the product purchased by a type 2 consumer providing that [math]p < p_0[/math].
[math]c[/math] = a positive constant
[math]Q[/math] = the level of the quality of the product

Type 2 consumers are very quality sensitive. However, type 2 consumers are insensitive to price. They will buy more of the product as the quality of the product increases regardless of the product's price. Nevertheless, all consumers do face some budgetary restrictions. We will, therefore, assume that this relationship only holds for
where \( q \). When the price per unit exceeds \( p_0 \), type 2 consumer will stop buying the product.

C. Market Behavior with Type 1 and Type 2 Consumers

We can study market behavior when the market consists of both type 1 and type 2 consumers. If the number of type 1 and 2 consumers is \( n_1 \) and \( n_2 \), respectively, then the behavior of the market will be described by equation (3).

\[
q_{12} = n_1 q_1 + n_2 q_2
\]  

(3)

where \( q_{12} \) = the total quantity of the product sold in a market consisting of both type 1 and type 2 consumers  
\( n_1 \) = the number of type 1 consumers  
\( n_2 \) = the number of type 2 consumers

Suppose a company wanted to sell a product in a market consisting of type 1 and type 2 consumers. The profit margin for this company would be the product's price minus the cost of adding quality to the product. In order to simplify the exposition we will assume that the cost function is quadratic. Our conclusions rely only on the assumption of a convex cost function. This assumption implies that as the level of quality increases it becomes more and more costly for the company to obtain additional quality. Eventually perfection is, indeed, difficult to achieve.

With a quadratic cost function, it is in the company's own best interest to adopt the price and the level of the quality given by equations (4) and (5), respectively.

\[
p = \frac{a}{2b} + \frac{3}{2} q^2
\]

\[
q = \frac{n_2 c}{2n_1 b}
\]

(4)

(5)

where \( p \) = the price charged for the product by the company  
\( q \) = the level of the quality of the product provided by the company if the market consists of type 1 and type 2 consumers

If the market consists of different geographic market segments with each segment consisting of different mixtures of type 1 and type 2 consumers, equation (4) would provide the price-quality relation that we find as we travel through the market. That relationship is depicted in figure 1.

As we see from figure 1, the price-quality relationship is nonlinear. (This result is consistent with the empirical work of McConnell, 1968.) At low prices, small changes in price correspond to large changes in quality. At higher prices, small changes in price correspond to smaller changes in quality. In all cases, however, higher prices correspond to higher levels of the quality. It is important to remember that we observe a price-quality relationship in the absence of competition between the geographic market segments. Hence, even without price competition, price levels still reflect the levels of the quality.

We can also determine how quality will vary from one geographic segment to another geographic market segment. Equation (5) tells us that as the relative number of type 1 consumers in the segment increases, the level of the quality decreases. Contrarily, as the relative number of type 2 consumers increases, the level of the quality increases. Acting in its own best interest, the company will decrease the product's price as the number of price sensitive (type 2) consumers increase. These relationships are depicted by figure 2 and figure 3.

C. Type 3 Consumers

In section IIB, we assumed that all consumers who were concerned with the quality of the product were also able to identify that quality. Some consumers, however, may want the quality but these consumers may be unable to determine whether or not the quality is present. We refer to these consumers as type 3 consumers.

We must use care in describing type 3 consumers. It could be argued that a consumer has no need for a quality of the product that the consumer cannot detect. For example, if a consumer cannot discriminate between an ordinary meal and a gourmet meal, we might wonder why that consumer would desire a gourmet meal. Of course, we could argue that the consumer is unfamiliar with the product and cannot easily inspect the product to determine the level of the quality. But this argument is not very compelling after the consumer has had the opportunity to use the product. At that point, the consumer would have learned the level of the quality. That information, then, could be used in future purchase decisions. Hence, the unfamiliarity argument could only be used for a new product or a durable good whose sales depend mainly on first purchases.

Nevertheless, there remain circumstances when a consumer may desire a quality which the consumer cannot detect. One circumstance might occur when the consumer is only a buyer for another consumer who can detect the quality. For example, a host may be purchasing wine for a party where guests are more discriminating than the host. Another example would be a generous individual who desires to provide a gift for another individual when the gift receiver possesses far more expertise about the gift than the gift giver. Finally, a consumer might desire a nutritious food product but the consumer may be unable to determine the nutritiousness of a specific food product.
Assuming that type 3 consumers exist, these consumers might use the level of price as a surrogate measure of the level of quality. We describe the behavior of type 3 consumers with equation (6).

\[ q_3 = d \]  
(6)

where \( q_3 \) = the quantity of the product purchased by a type 3 consumer providing \( p < p_0 \)

\( d \) = a positive constant

Again, type 3 consumers do eventually face a budget constraint, so we assume equation (6) only holds for a price less than \( p_0 \). At prices exceeding \( p_0 \), type 3 consumers will not purchase the product. At prices less than \( p_0 \), type 3 consumers behave contrary to typical economic behavior.

Suppose the number of type 3 consumers is \( n_3 \). In this situation, a company seeking its own best interest will set its price and quality according to equations (7) and (8), respectively. (See the appendix for details.)

\[ p = \frac{an_1}{2bn_1 - 2dn_3} + \frac{3}{2} Q^2 \]  
(7)

\[ Q = \frac{n_3 c}{2bn_1 - 2dn_3} \]  
(8)

where \( n_3 \) = the number of type 3 consumers.

We see that as the number of type 3 consumers increases, the price charged for the product increases. We also see that as the number of type 3 consumers increases, the level of the quality also increases. Even though these consumers cannot detect quality, their desire for quality causes the company to increase the level of the quality.

This conclusion may seem, at first glance, somewhat counterintuitive. We might question how consumers, who cannot detect quality, can encourage the company to provide more quality. Upon reflection, however, the answer is simple. Without type 3 consumers, i.e., \( n_3 = 0 \), the company is selling its product to both type 1 and type 2 consumers. The quality-sensitive type 2 consumer wants quality at almost any cost. The company would provide high quality at a high price. If it were not for the price-sensitive type 1 consumer, these price-sensitive consumers depress the price and, because they place no value on quality, they also depress quality levels. With the introduction of type 3 consumers, the effect of type 1 consumers is diminished. With this effect diminished, the company is pleased to raise the price of the product while simultaneously persuading quality-sensitive type 2 consumers to increase their purchases by increasing the quality of the product.

We should note that type 3 consumers could vastly outnumber type 2 consumers and the integrity of equations (7) and (8) would not be impugned. Only when the number of type 3 consumers exceeds the number of type 1 consumers would the price-quality relationship be destroyed. Hence, even when very few consumers are able to detect quality, price-quality relationships exist. It is only when few price sensitive consumers exist that price-quality relationships are destroyed. This fact may explain why even when we find few consumers can detect quality, companies still find drastic decreases in sales when they lower quality levels.

III. The Effect of Competitive Entry

In section II, we found that even without competition, a price-quality relationship exists. Moreover, we found that the existence of type 3 consumers, who use price as a surrogate measure of quality, actually cause the level of the quality to increase. In this section, we examine the effect of competition on the price-quality relationship.

A. High-Priced Low-Quality Competition

In section II, we found the level of the price and the level of the quality that a company would adopt if there were no competitive brands in the market. These levels are given by equations (7) and (8). Substituting the quality level given by equation (8) into equation (7), we obtain the level of price given by equation (9).

\[ p = \frac{an_1}{2(bn_1 - dn_3) + \frac{3}{8}(n_3 c)^2} \]  
(9)

Equation (9) shows that the product's price depends on the number of each type of consumer in the market and the behavior of some average consumer. It is the mixture of these consumers which creates the price-quality relationship. However, suppose a second company introduced a high-priced, low-quality brand into the market. We might wonder if that introduction would destroy the price-quality relationship we previously observed across geographic market segments.

The introduction of a high-priced, low-quality brand would attract type 3 consumers. Type 3 consumers would be attracted to the new brand because they would incorrectly associate the new brand's high price with a high level of the quality. We see from equation (8) that a decrease in \( n_3 \) results in a lower quality for the new brand. Equation (9) illustrates that a decrease in \( n_3 \) results in a decrease in the price of the old brand. Hence, the original company will respond to this competitive entry by lowering both its price and its quality.

B. Other Forms of Competition

In section IIIA, we investigated the impact of the entry of a high-priced, low-quality competitive brand. However, other types of brands can also enter the market. A competitor might introduce a low-priced, low-quality brand. In this case, the original company would lose some fraction of the type 1 consumers to the new brand. We see from equation (8) that the original company would respond by increasing the level of the brand's quality as \( n_3 \) decreases. Equation (9) tells us that the original brand's price will increase as the number of type 1 consumers decreases because of losses to the new brand.

Another possible competitive entry might be a low-priced, high-quality brand. In this case, the original company would lose some fraction of type 1 and type 2 consumers to the new brand. The original company will change its quality level to that level given by equation (10).

\[ Q = \frac{n_3 c}{2bn_1 - 2dn_3} \]  
(10)

where \( r = \) the fraction of type 1 and type 2 consumers lost to the new brand. Note that \( 0 < r < 1 \).

Equation (10) indicates as type 1 and 2 consumers are lost to the new brand, the old brand will increase the old brand's quality. The corresponding price level is given by equation (11).

\[ p = \frac{an_1}{2bn_1 - 2dn_3 + \frac{3}{8}(n_3 c)^2} \]  
(11)

We see from equation (11) that as type 1 and type 2 consumers move from the old brand to the new brand, i.e.,
as $r$ increases, the old brand will raise its product's price. We see that as the fraction of lost consumers increase, the original company responds by increasing both its level of quality and its level of price.

Finally, the introduction of a high-priced, high-quality competitive brand, would result in the movement of type 2 and type 3 consumers from the old brand to the new brand. We see from equation (8) that a decrease in the number of type 2 and type 3 consumers results in a decrease in the quality of the old brand. We see from equation (9) that as $n_2$ and $n_3$ decrease, the old brand's price also decreases. The results of this section are summarized in Table 1.

### TABLE 1

<table>
<thead>
<tr>
<th>Competitive Entry</th>
<th>High Quality</th>
<th>Low Quality</th>
</tr>
</thead>
<tbody>
<tr>
<td>High Priced</td>
<td>Lower Price</td>
<td>Lower Price</td>
</tr>
<tr>
<td></td>
<td>Lower Quality</td>
<td>Lower Quality</td>
</tr>
<tr>
<td>Low Priced</td>
<td>Raise Price</td>
<td>Raise Price</td>
</tr>
<tr>
<td></td>
<td>Raise Quality</td>
<td>Raise Quality</td>
</tr>
</tbody>
</table>

In every case, price goes in the same direction as the level of quality.

IV. Determining the Quality Brand

Suppose a second company were to enter the market of the first company and offer a second product which does not possess the quality of the first company's product. Further, suppose that the second company offers this second product at the same price as the first company's product. We already have examined how the first company should react to this new competition. However, suppose the second company continues to match the price of the first company without, of course, providing the associated quality. We might wonder if a consumer who cannot identify the quality can ever distinguish between the two products.

The answer to this query is related to the advertising expenditures of the two companies. We might argue that the first company could use advertising to alert type 3 consumers of the quality differential. With informative advertising, type 3 consumers would no longer need to use the price as a surrogate for the level of quality of the product. However, we could argue that the second company could use deceptive advertising to falsely claim that the second company's product also had the quality. If this counter advertising were to occur, type 3 consumers, who are unable to determine quality from inspection, would be unable to use the advertising message as a mechanism to distinguish the products.

Nelson (1974) argues that the second company does not have the incentive to advertise in this way because advertising is an investment. Advertising expenditures, Nelson argues, must be justified not only on the basis of added current sales but also on the basis of additional future sales. Hence, companies with inferior products will not be willing to make large investments in advertising because that advertising can only increase current sales. Once product trial has occurred, consumers will realize that the product is inferior and future sales will not be improved through advertising. Only companies with superior products will be willing to invest heavily in advertising because only these companies can use advertising to provide a mechanism to distinguish the products.

Nelson's argument, of course, fails if type 3 consumers exist. His argument is compelling as long as consumers learn. Learning prevents making a mistake twice and, hence, encourages only companies who expect repeat purchases to invest heavily in convincing consumer to make an initial purchase. Type 3 consumers never learn so encouraging initial purchases by these consumers does not preclude future purchases even when the product does not possess the quality.

Rather than making the tautological argument that the company with the quality product will have the convincing advertising, we will merely allow a company's advertising to make consumers aware of the company's brand. Therefore, the company offering the quality product will sell the quantity of its product given by equation (12).

We arrive at equation (12) by multiplying the quantity demanded given 100% awareness by the fraction of consumers aware of the product. For expositional purposes, we will assume that awareness levels are quadratically related to advertising expenditures. That is, it takes $A$ squared dollars to achieve an awareness level of $A$. In order to simplify the mathematics, we further assume that type 1 and type 3 consumers are divided evenly between the two brands, i.e., $R = 1/2$. The result is equation (13).

\[
q_{123} = \left( n_1 q_1 + n_2 q_2 \right) \left( 1 - R \right) + n_2 q_2 A
\]

where $q_{123} = \text{the quantity of the quality brand sold in a market consisting of all three types of consumers;}$

$A = \text{the fraction of consumers aware of the quality brand;}$

$R = \text{the fraction of consumers lost by the quality brand to the new brand which does not possess the quality.}$

Equation (13) tells us that advertising expenditures will increase as the level of quality increases. Furthermore, the brand without the quality spends the least on advertising. Hence, advertising expenditures, i.e., the amount of advertising by a company, is a positive indicator of the quality level of the company's brand.

Equation (13) does assume an optimal price. Unfortunately, if the second company were to match the price of the first company, the result might be a non-optimal price for the second company. Hence, for completeness, we compute the advertising expenditures of the second company when it matches prices instead of setting an optimal price. The resulting expenditures are given by equation (14). We see that this alternative computation does not change our previous conclusions. (See the appendix for details.)

\[
E_1 = \frac{a n_1 + (b n_1 - d n_3)}{256 (b n_1 - d n_3)^2}
\]

where $E_1 = \text{the level of advertising expenditures for the company with the brand having the quality.}$

Equation (13) does assume an optimal price. Unfortunately, if the second company were to match the price of the first company, the result might be a non-optimal price for the second company. Hence, for completeness, we compute the advertising expenditures of the second company when it matches prices instead of setting an optimal price. The resulting expenditures are given by equation (14). We see that this alternative computation does not change our previous conclusions. (See the appendix for details.)

\[
E_2 = \frac{a^2 n_1^2 - 9(b n_1 - d n_3)^2}{256 (b n_1 - d n_3)^2}
\]

where $E_2 = \text{the level of advertising expenditures for the company whose brand does not have the quality;}$

$Q = \text{the level of quality of the other brand.}$

The mathematics, of course, provides direct implications, but the intuition behind the mathematics is also important. Equation (14) shows that the company with the quality brand will spend more on advertising because of the existence of
type 2 consumers. These type 2 consumers provide a potential market for the quality brand which does not exist for the product without the quality. With this added potential market, the return from advertising is greater for the company with the quality brand. Finally, with a greater return from advertising, it is in the best interests of the company with the high quality brand to spend more on advertising.

V. The Two Quality Product

In sections II, III and IV, we examined consumer and company behavior toward a product with one quality. However, product quality may be multidimensional. Some consumers may be interested in one quality of the product while other consumers may be interested in another quality of the product. For example, some consumers may judge the quality of an automobile in terms of its luxuriousness while other consumers may judge the quality of the automobile in terms of its need for repairs. In this section, we investigate products which have two qualities.

A. Type 4 Consumers

Some consumers may be interested in a different quality of the product than the quality of the product desired by type 2 consumers. Again, we will not require type 4 consumers to behave according to the economic model of behavior. We allow these consumers to seek the quality that they desire, regardless of the product's price. These individuals are type 4 consumers. We describe the behavior of type 4 consumers with equation (15).

\[ q_4 = e Q_2 \]  

\[ q_4 = \text{the quantity of the product purchased by a type 4 consumer providing } p < p_0; \]
\[ e = \text{a positive constant; } \]
\[ Q_2 = \text{the level of the second quality of the product.} \]

Type 4 consumers are very quality sensitive. Neither type 4 consumers nor type 2 consumers are price sensitive. Nevertheless, at some point, a budgetary constraint becomes important. We, therefore, assume that once the product's price exceeds \( p_0 \), both type 2 and type 4 consumers will stop purchasing the product.

Type 2 and type 4 consumers do differ on the quality they desire. Each type of consumer wants a different quality of the product to the exclusion of the other quality. Although we might find some correlation between the levels of the two qualities, we assume that the company has the capability of offering each quality at a different level.

B. Market Behavior with Type 1, 2, 3 and 4 Consumers

We can study market behavior when the market consists of type 1, 2, 3 and type 4 consumers. If a product were offered in this market, the quantity of that product which would be sold is given by equation (16).

\[ q_{1-4} = n_1 q_1 + n_2 q_2 + n_3 q_3 + n_4 q_4 \]  

\[ q_{1-4} = \text{the total quantity of the product sold in a market consisting of type 1 through type 4 consumers; } \]
\[ n_4 = \text{the number of type 4 consumers.} \]

Suppose a company wanted to sell this product. The profit margin for this company would be the product's price minus the cost of adding both qualities to the product. In order to simplify the exposition, we will again assume that the cost function for each quality is quadratic.

With a quadratic cost function, it is in the company's own best interest to adopt the price and levels of the qualities given by equations (17), (18) and (19).

\[ p = \frac{2b n_1 - 2n_2 + \frac{1}{2} c_1^2 + \frac{3}{2} c_2}{2n_2} \]  

\[ q_1 = n_2 c/(2n_1 b - 2n_3 d) \]  

\[ q_2 = n_4 e/(2n_1 b - 2n_3 d) \]

If the market consists of different geographic market segments with different mixtures of consumer types, we would observe a price-quality relationship. The relationship, however, is more complex than the relationship discussed in section IIC. In this case, the price represents the combined effect of both qualities of the product. It is impossible to infer the level of both qualities from the product price. Equation (17) can only be used to find \( q_1 + q_2 \).

Suppose the number of type 3 consumers increases. In section IIC, we found that these price-seeking consumers caused an increase in the level of the quality. In this case, equations (18) and (19) tell us that an increase in the number of type 3 consumers increases the levels of both product qualities. Both qualities increase but in a way that the proportion \( Q_1/Q_2 \) remains constant.

If we travel from one geographic market segment to another geographic market segment which contains different numbers of type 2 and type 4 consumers, the price will reflect combined quality levels. In order to determine the level of each quality, it would be necessary to know the ratio of type 2 to type 4 consumers. If quality 1 is valued, a large ratio is desired. If quality 2 is valued, a small ratio is desired. This result means a consumer can use price as a measure of the desired quality if that consumer's values are reflected by other consumers in the market. If the consumer desires the first quality, the consumer can only use price as a surrogate for quality when the number of type 2 consumers is large. If the consumer desires the second quality, the consumer can only use price as a surrogate for quality when the number of type 4 consumers is large.

VI. Summary and Conclusion

From the preceding sections, we come to the following conclusions.

1. Prices reflect levels of quality even with limited competition.
2. The quality-price relationship is non-linear.
3. Prices reflect levels of quality even when some consumers do not behave in a rational economic manner.
4. Consumers using price as a surrogate measure of quality encourage companies to raise the level of product quality.
5. Competition does not destroy the relationship between price and quality.
6. Companies with high quality products spend more on advertising than companies offering lower quality products.
7. When different qualities of the product are important, price can only be used as a measure of the quality desired by the market. In other words, a consumer can only use price as a measure of quality if the consumer's values are reflected by other consumers in the market.

The preceding conclusions provide some implications for future research in consumer behavior. When determining
when consumers use price as a cue to quality, we now know some other cues which consumers might use in conjunction with price. For example, we might ask the following questions.

- Do consumers compare their preferences with other consumer preferences when using price as a cue to quality?
- Do consumers use advertising and price cues simultaneously? Do they use the existence of advertising or the advertising message?
- Do consumers perceive a non-linear price-quality relationship as depicted by Figure 1?
- Do consumers use the degree of competition in the market in conjunction with price as a cue to quality?

Our message to consumers is: "You get what you pay for when what you want, is what others want!"

References


Appendix

Derivation of Equations (4) and (5)

See the derivation of equations (4) and (5) while substituting zero for $n_3$.

Derivation of Equations (7) and (8)

To maximize $(p - Q^2)q_{123}$ we take the derivatives with respect to $p$ and $Q$ and set these derivatives equal to zero. That action yields equations (A-1) and (A-2).

$$q_{123} = (b_1 - dn_3)(p - Q^2)$$  \hspace{1cm} (A-1)

$$-2Qq_{123} = -cn_2(p - Q^2)$$  \hspace{1cm} (A-2)

These equations (taking the ratio) imply equation (7), that is, $2Q = cn_2/(b_1 - dn_3)$. Substituting this value for $Q$ into $q_{123}$, we find that $q_{123} = an_1 - (b_1 - dn_3)p + (c_2^2n_2^2)/(2bn_1 - 2dn_3)$. Rearranging, finds $q_{123} = an_1 - (b_1 - dn_3)p + 2(bn_1 - dn_3)Q^2$. Substituting this value for $q_{123}$ into equation (A-1) yields equation (A-3).

$$an_1 - (b_1 - dn_3)p + 2Q^2 = p - Q^2$$  \hspace{1cm} (A-3)

Solving equation (A-3) for $p$ yields equation (8).

Derivation of Equation (13)

To maximize $(p - Q^2)q_{123} - A^2$, we take the derivative with respect to $A$ and set it equal to zero which yields equation (A-9). (Remember $R = 1/2$)

$$2A = (p - Q^2)(n_1q_1 + 2n_2q_2 + 3n_3q_3)/2$$  \hspace{1cm} (A-4)

Simplifying, we obtain equation (A-5).

$$2A = (p - Q^2)(n_1a - p(n_1b - n_3d) + 2n_2cQ)/2$$  \hspace{1cm} (A-5)

Rearranging, we obtain equation (A-6) because $Q = n_2c/(b_1 - dn_3)$ when $R = 1/2$.

$$2A = (p - Q^2)(K - p + 2Q^2)(n_1b - n_3d)/2$$  \hspace{1cm} (A-6)

where $K = n_1a/(n_1b - n_3d)$

From equation (7), we know $p = (K + 3Q^2)/2$. Substituting this value for $p$ into (A-6) yields (A-7).

$$2A = (K + 3Q^2)^2(n_1b - n_3d)/8$$  \hspace{1cm} (A-7)

Now, $E_1 = A^2$, hence, equation (A-8) follows.

$$E_1 = (K + Q^2)^2(n_1b - n_3d)^2/256$$  \hspace{1cm} (A-8)

Substituting for $K$ yields equation (13).

Derivation of Equation (14)

To maximize $(p - Q^2)q_{123} - A^2$, we take the derivative with respect to $A$ and set it equal to zero which yields equation (A-9).

$$2A = (p - Q^2)(n_1q_1 + 2n_2q_2 + 3n_3q_3)/2$$  \hspace{1cm} (A-9)

Substituting $p = (K + 3Q^2)/2$, we obtain (A-10).

$$2A = (K + 3Q^2)(2n_1q_1 - (b_1 - dn_3)(K + 3Q^2))/8$$  \hspace{1cm} (A-10)

Simplifying, we obtain equation (A-11).

$$2A = (K + 3Q^2)(b_1 - dn_3)/8$$  \hspace{1cm} (A-11)

Noting that $E_1 = A^2$ and substituting for $K$ yields equation (14).