Product Assortment In A Triopoly
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Abstract:
In many industries, the producers of higher quality products offer smaller assortments of flavors, colors, sizes, etc. When and why producers of super-premium products should find it profitable to offer a smaller assortment than the makers of non-premium products is explored by deriving a Nash equilibrium on prices and product assortments for a Triopoly. This Triopoly includes producers of a super-premium, a regular, and an economy ice cream. It is assumed that an ice cream producer's sales increase as that producer increases its assortment and that the economy producer competes more with the regular producer than with the super-premium producer. The results show that, at the Nash equilibrium, additional product quality, increased consumer price sensitivity, and greater assortment costs discourage product assortment. A larger market potential, sharper competition, and greater competitive costs encourage more assortment by the super-premium producer.
PRODUCT ASSORTMENT IN A TRIOPOLY*

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Producers of super-premium ice cream, such as Häagen-Dazs, offer a smaller assortment of flavors than the producers of lesser quality ice cream. Examples of this phenomenon can be found in other industries as well. In many industries, the producers of higher-quality products offer a smaller assortment of flavors, colors, sizes, patterns, textures, fragrances, tones, styles, models, designs, types or other options. This paper explores when and why producers of super-premium products should find it profitable to offer a smaller assortment than the producers of nonpremium products. We derive a Nash equilibrium both on prices and product assortments for a triopoly. At the Nash equilibrium, we show that additional product quality, increased consumer price-sensitivity and greater assortment costs discourages product assortment. We also show that a larger market potential, greater competitive costs and sharper competition encourage more assortment by the super-premium producer.

(introduction)

Both the marketing and economics literature consider product differentiation among competing firms (e.g., Urban and Hauser 1980; Mussa and Rosen 1978; Spence 1980). These literatures concentrate on the degree and nature of differentiation occurring among competitive products (Lancaster 1980; Ratchford 1975). Most of this research considers how a producer of a product should differentiate that product (Hauser and Simmie 1981; Sen 1982), how current products are positioned (Gavish, Horsky and Srikanth 1983; Shugan 1987), and how differentiated products should be introduced (Kalish and Lilien 1986; Shocker 1974). However, empirical observation reveals that nearly every producer offers a mix or line of products. Producers of durables (e.g., automobile), nondurables (e.g., linens), industrial products (e.g., carpets) and services (e.g., interior design) all offer product lines with a wide assortment of items. Recent research does consider product lines. For example, recent research has considered pricing a product line (Monroe and Zoltners 1979; Oren, Smith and Wilson 1984; Reibstein and Gatignon 1984; Shugan 1984), product line structuring (Green and Krieger 1985; McBride and Zufryden 1988, Shugan and Balachandran 1977; Urban 1969; Zufryden 1982), retail pricing of product lines (Shugan 1986) and the effect of product popularity on the distribution of product lines (Hanson 1986). However, this research does not address the problem of the appropriate size or assortment of a product offering.

This paper considers the amount of product assortment that should be offered by a producer. We concentrate our effort on product assortment in the ice cream industry. Product lines in this industry have three important characteristics. First, ice cream product lines usually offer an assortment of flavors. The breadth of this assortment can vary from a handful of flavors to over 50 flavors. Second, nearly all flavors within a product line are offered at the same price. Although a few flavors are sometimes offered at a premium price, the majority of flavors are sold at the same price. Finally, although different ice cream lines have different qualities, ice cream quality is often quite similar within a line.

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1 Our references to ice cream refer to hard frozen ice cream.
of flavors. Unlike a camera producer which might offer a camera line ranging from an economy starter model to a high-quality professional model, most ice cream lines usually show a low variance in quality. For this reason, we can typically refer to the quality of the entire ice cream line. For example, in Chicago, Häagen-Dazs Natural Ice Cream is a high-quality or super-premium ice cream line while Bresler's 33 Flavors is an economy line. Unlike economy ice cream, all Häagen-Dazs super-premium ice cream provides more butterfat content (about 18% rather than 10%), less air content (only 20% rather than 50%), more natural ingredients (e.g., real fruit rather than artificial flavors), more expensive sweeteners (e.g., honey and real sugar), and more expensive containers. Hence, within an ice cream line, a consumer can often choose from an assortment of flavors, each having the same quality and the same price. Also, the unit variable cost of producing a higher quality ice cream is greater than the unit variable cost of producing a lower quality ice cream.

This paper takes advantage of these three special characteristics of most ice cream assortments. Not all industries offer assortments of items which are all of the same quality and price. As stated earlier, some product-lines offer a range of qualities where the price of each item varies according to the level of quality. However, many industries do offer product lines which have an assortment of merchandise at the same price (e.g., Parsons 1985). Shoes, shirts and hats have assortments of sizes. Toiletries, deodorants and bathroom sprays have assortments of fragrances. Doorbells and clocks have assortments of chimes. Other industries offer assortments of colors, motifs, designs, flavors, styles, awards, types, patterns, textures and so on. Hence, our analysis may be applicable to industries beyond ice cream.

Within the ice cream industry, producers of super-premium ice creams often have less of an assortment of flavors than producers of economy ice creams (Kochak 1985). For example, Pillsbury's subsidiary Häagen-Dazs, with sales of about $100 million, offers far fewer flavors than Bresler's 33 Flavors. See Table 1. Bresler's offers no super-premium ice creams. Moreover, producers of both regular flavors and super-premium flavors offer a much greater assortment of regular flavors than super-premium flavors. For example, in 1985, Baskin-Robbins, with 3200 units worldwide, offered only two flavors of super-premium ice cream (French Vanilla and Chocolate Fudge) and had no plans to offer additional super-premium flavors (Kochak 1985). Hence, flavor assortments for ice cream lines are often negatively correlated with line quality.

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2 "Häagen-Dazs Sticking it to Dove," *Ad Week*, 27 (February 1986), 2.
3 Ice cream is one of three products which does not have to list the use of artificial coloring on the package.
Clearly, increased assortment costs cause less assortment. Sufficiently large assortment costs can explain all observed assortments. We want to go beyond this obvious observation. We want to fully understand the conditions causing a negative correlation between product quality and assortment. We proceed as follows. First, we present a triopoly model where each of three producers sets both price and product assortment. Next, we derive the optimal assortments at the Nash Equilibrium. Then we answer three questions: Which factors decrease optimal assortment? Which factors decrease relative assortment? Should Häagen-Dazs offer fewer flavors? Finally, we summarize our results and offer our conclusions.

Our Model and Our Assumptions

To specify when a super-premium producer should offer less assortment, we make several simplifying assumptions. First, we posit three ice cream producers in the market who adopt direct distribution. We denote these producers as \( i, j \) and \( k \). Without loss in generality, denote producer \( i \) as a super-premium producer, \( j \) as a regular producer and \( k \) as an economy producer. For example, in Chicago, there are three dominant producer-owned outlets of hard frozen ice cream: Häagen-Dazs Natural Gourmet Ice cream, Baskin-Robbins 31 Flavors, and Bresler’s 33 Flavors. We choose three producers because three is the minimum complexity that still allows examination of the correlation between product quality and product assortment. Our analysis could be extended beyond a triopoly to a larger number of producers. However, we believe that modelling a larger number of producers does not provide sufficient incremental insight to justify the additional complexity.

We also posit a constant elasticity demand function for each of the three producers. Although this demand function represents only one of many possible demand functions, this demand function has several favorable properties (Lilien and Kotler 1983, p. 73). The functional form of the constant elasticity demand function (also called the multiplicative model, C.E.S. function or log-linear specification) has seen applications in both empirical and theoretical research (e.g., see Chiang 1974; Intriligator 1978, p. 220; Krishnamurthi and Raj 1985; Pindyck and Rubinfeld 1981, p. 107; Ratchford 1982; Reibstein and Gatignon 1984). The constant elasticity demand function is almost as tractable as the linear demand function yet the constant elasticity demand function has several useful properties for the application in this paper. For example, a linear demand function implies that a price increase from $1 to $2 will cause the same absolute decrease in sales as a price increase from $100 to $101. This property of the linear demand function would not be desirable for the application in this paper. Most ice cream producers believe that the absolute sales loss caused by a $.25 price increase is greater for a low-priced ice cream than a high-priced ice cream. Nevertheless, we will discuss extensions to a linear demand curve in a subsequent section.

Finally, a large body of literature deals with the estimation of the constant elasticity demand function because it is intrinsically linear. Although this paper is only concerned with a theoretical investigation of product assortment, future research should certainly be concerned with empirical investigations. It is comforting to know that our functional forms are not only mathematically tractable, but estimable as well.

Given a constant elasticity demand function, the unit sales of producers \( i, j \) and \( k \) are given by equations (1), (2) and (3), respectively. The symbol “\( = \)” denotes “is defined as”.

\[
q_i(p_i, p_j, p_k, s_i) = \alpha_i p_i^{\beta_i} p_j^{\gamma_i} p_k^{\delta_i} s_i(\sigma_i, \sigma_j, \sigma_k),
\]  

(1)

\footnote{To consider retail distribution, we must model the retailer.}
\[ q_i(p_i, p_j, p_k, s_i) = \alpha_i p_i^{\beta_i} p_j^{\delta_i} p_k^{\gamma_i} s_i(\sigma_i, \sigma_j, \sigma_k), \]

(2)

\[ q_k(p_i, p_j, p_k, s_k) = \alpha_k p_i^{\beta_k} p_j^{\delta_k} p_k^{\gamma_k} s_k(\sigma_i, \sigma_j, \sigma_k), \]

(3)

\[ q_i = \text{the unit sales of producer } i \text{'s product line when the price of items in producer } i \text{'s product line is } p_i \text{ and the competitor's prices are } p_j \text{ and } p_k, \text{ respectively.}
\]

\[ q_j = \text{the unit sales of the product line of the regular producer, } j.\]

\[ q_k = \text{the unit sales of the product line of the economy producer, } k.\]

\[ p_r = \text{the price of the product line of producer } r; \ r \in \{i, j, k\}.\]

\[ \alpha_r = \text{a measure of the producer } r \text{'s absolute market potential; } r \in \{i, j, k\}.\]

\[ s_r = \text{a measure of the comparative assortment of producer } r \text{'s product line compared with the assortments of competitive product lines; where } r \in \{i, j, k\}\).\]

\[ \beta_{rr} = \text{the own-price elasticity of demand for producer } r; \text{ where } \beta_{rr} > 1 \text{ and } r \in \{i, j, k\}. \]

\[ \beta_{rt} = \text{producer } r \text{'s cross price elasticity of demand with respect to producer } t \text{'s price; where } \beta_{rt} < 1 \text{ and } r, t \in \{i, j, k\} \text{ and } r \neq t. \]

The variable \( s_i \) measures producer \( i \)’s comparative assortment. Without loss in generality, scale \( s_i \) so that the comparative assortment elasticity of demand is one. We implicitly assume that an ice cream producer’s sales increase as that producer increases its assortment compared with competitive assortments. The increased sales could come from increased distribution (Shugan 1983), from new customers entering the market, from customers of competitive ice creams or from variety-seeking by existing customers (Jeuland 1978; Givon 1984). Finally, we assume that all of the previously defined variables are strictly positive and that the economy producer competes more with the regular producer than the super-premium producer (i.e., \( \beta_{jk} > \beta_{ik} \)).

Applying equations (1), (2) and (3) requires expressions for the comparative assortments. We posit that comparative assortments are linear functions of the absolute assortments. This assumption implies that interaction effects between own assortment and competitive assortment are sufficiently small so that interactions can be ignored.\(^7\) In other words, we assume that the variables, own price, own assortment, competitive price and competitive assortment, overwhelm the higher order interaction effects and make our approximation sufficiently descriptive of the real market. The burden of our assumption is partially relieved by allowing the absolute producer assortments to be arbitrarily scaled. Given this assumption and absolute producer assortments of \( \sigma_i, \sigma_j \) and \( \sigma_k \), the comparative assortments are given by equations (4), (5) and (6), respectively.

\[ s_i(\sigma_i, \sigma_j, \sigma_k) = (\omega_i \sigma_i + \omega_j \sigma_j + \omega_k \sigma_k), \]

(4)

\[ s_j(\sigma_i, \sigma_j, \sigma_k) = (\omega_i \sigma_i + \omega_j \sigma_j + \omega_k \sigma_k), \]

(5)

\[ s_k(\sigma_i, \sigma_j, \sigma_k) = (\omega_k \sigma_i + \omega_j \sigma_j + \omega_k \sigma_k), \]

(6)

\[ \omega_r = \text{a real constant reflecting the impact of producer } r \text{'s assortment on the comparative assortment of producer } r; \text{ where } \omega_r > 0, r \in \{i, j, k\}. \]

\(^7\) Our objective is to study assortments rather than prices. Therefore our treatment of product assortment is more complex than our treatment of price. For example, when the price of producer \( j \) doubles, the demand for producer \( i \) increases by the factor \( 2^{0_j} \). In contrast, when the assortment of producer \( j \) doubles, the demand for producer \( i \) increases by the factor \( (\omega_j \sigma_j + 2\omega_j \sigma_j + \omega_k \sigma_k)/(\omega_i \sigma_i + \omega_j \sigma_j + \omega_k \sigma_k). \) This later factor depends on the absolute assortments of all producers in the market. In contrast to changes in price, the effect of producer \( j \)’s absolute assortment on producer \( i \)’s demand depends on the absolute assortment of producer \( i \).
### Table 2

**Ice Cream Flavors as Percent of Sales**

<table>
<thead>
<tr>
<th>Flavor</th>
<th>Percentage of Sales</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Vanilla</td>
<td>35.1%</td>
</tr>
<tr>
<td>2. Chocolate</td>
<td>12.4%</td>
</tr>
<tr>
<td>3. Neapolitan</td>
<td>7.4%</td>
</tr>
<tr>
<td>4. Chocolate Chip</td>
<td>5.9%</td>
</tr>
<tr>
<td>5. Strawberry</td>
<td>5.6%</td>
</tr>
<tr>
<td>6. Vanilla Fudge</td>
<td>4.2%</td>
</tr>
<tr>
<td>7. Butter Pecan</td>
<td>2.7%</td>
</tr>
<tr>
<td>8. Cherry</td>
<td>2.5%</td>
</tr>
<tr>
<td>9. Butter Almond</td>
<td>1.6%</td>
</tr>
<tr>
<td>10. French Vanilla</td>
<td>1.4%</td>
</tr>
</tbody>
</table>

Source: International Association of Ice Cream Manufacturers

Equations (4) through (6) define each producer’s comparative assortment, in terms of the absolute assortments of all three producers. Although our definition of comparative assortment cannot represent all possible relationships between absolute producer assortments and individual producer demands, it can represent a variety of possible relationships. The constants, $\omega_{ij}$ for $r \neq t$, in equations (4) through (6) may be either positive or negative, depending on the market situation. For example, when $\omega_{ij}$ is negative, additional assortment by producer $j$ hurts producer $i$’s sales. Perhaps, increases in producer $j$’s assortment cause producer $i$’s customers to switch to producer $j$. When $\omega_{ij}$ is positive, additional assortment by producer $j$ helps producer $i$’s sales. Perhaps, producer $j$’s additional assortment attracts new consumers into the product category, thereby helping producer $i$’s sales.

There are three approaches to measuring absolute producer assortments. The first approach uses some function of the number of items in the producer’s product line. For ice cream, product assortment becomes a function of the number of flavors. For other products, assortment becomes a function of the number of colors, models, sizes, patterns, textures, fragrances, tones, styles, and so on. The function, measuring assortment, should be concave. This functional shape is consistent with empirical observations showing that the top ten flavors accounted for 78.8% of all ice cream sales. See Table 2.

We can also measure producer assortment by examining the item attribute variability within the producer’s product line. In the case of ice cream, we might examine the differences among available flavors. For example, a three-flavor product line consisting of chocolate, chocolate chip and fudge might have less assortment than another three-flavor line consisting of chocolate, vanilla and strawberry. Finally, a third approach to measuring a product line’s assortment is the construction of a complex function whose arguments are both the number of flavors and the attribute variability of these flavors.

Before proceeding, we should consider the consequence of variety-seeking behavior (e.g., see Kahn, Kalwani and Morrison 1986) on product assortment. Much research (e.g., Lattin and McAlister 1985) shows the importance of variety-seeking behavior in many product categories. With variety-seeking behavior, consumers wish to purchase a current flavor which differs from the flavor purchased last period (e.g. see McAlister and Pessemier 1982). Hence, consumers might not consistently purchase from a producer.

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who does not offer a sufficient assortment to allow for switching flavors. The magnitude of this effect is captured by the variables \( \omega_{ir}, r, i \in \{i, j, k\} \), in equations (4), (5) and (6). It appears, therefore, that variety-seeking necessitates increased product assortment. However, ice cream producers can meet the demand for variety by changing flavors over time. For example, an ice cream producer might offer Amareto Cheesecake ice cream during October and offer Pumpkin Pie ice cream during November. Steve’s Ice Cream shops, an Integrated Resources subsidiary, makes 50 ice creams but only offers 12 flavors on a rotating basis (Kochak 1985). Hence, variety-seeking behavior does not necessarily imply a greater assortment of flavors.

**Optimal Assortments at the Nash Equilibrium**

From equations (1) through (6), it is obvious that the three producers \((i, j \text{ and } k)\) face symmetric conditions. Therefore, we restrict our attention to producer \(i\), i.e., the super-premium producer. Equation (7) depicts the super-premium producer’s profit (\(\Pi_i\)). Symmetric profit functions exist for producers \(j\) and \(k\).

\[
\Pi_i = \alpha_i(p_i - c_i)p_i^{-\gamma_i}p_j^{\beta_{ij}}p_k^{\beta_{ik}}(\sigma_i, \sigma_j, \sigma_k) - h_i\gamma_i \quad \text{where:} \quad (7)
\]

\(c_i\) = the unit variable cost of production for producer \(i\),
\(h_i\) = a real constant; \(h_i > 0\),
\(\gamma_i\) = a real constant; \(\gamma_i > 1\).

Equation (7) resembles most profit functions found in the marketing literature (e.g., see Eliashberg and Chatterjee 1985; Eliashberg and Jeuland 1983; Hauser and Shugan 1983; Hauser and Simmie 1981; Jeuland and Shugan 1983; McGuire and Staelin 1983). The product line’s profit margin is given by \(p_i - c_i\). The demand function obeys equations (1) through (4). However, unlike the existing marketing literature, we posit that increasing product assortment increases assortment costs at a nonlinear rate. Of course, the production literature recognizes that increased product assortment causes increased production costs. In fact, researchers in production encourage the process of “simplification” which attempts to reduce costs by reducing the number of types of products. Simplification reduces inventories, decreases investments in production equipment, improves planning, simplifies production methods, increases production rates, simplifies inspection, saves storage space, supports control, reduces required technical personnel, decreases the sales price and shortens order queues (Eilon 1962, p. 78). The constant \(\gamma_i\) depicts the rate at which costs increase with additional product assortment. We require \(\gamma_i > 1\) to insure a finite optimal assortment. The constant \(h_i\) depicts the absolute cost associated with additional product assortment.

Here, we treat assortment costs as production changeover costs. Sometimes, additional assortment might increase or decrease variable costs. For example, more assortment might imply raw material purchasing economies (decreased variable costs) or lower-volume less-economic runs (increased variable costs). We assume that the primary effect of adding assortment is on production set-up costs (fixed costs) rather than variable costs.

We differentiate \(\Pi_i\) with respect to \(p_i\) and \(\sigma_r, r \in \{i, j, k\}\), set all six derivatives equal to zero and simultaneously solve the six equations for the Nash equilibrium prices, denoted \(p_i^*, p_j^*, p_k^*\), and the Nash equilibrium assortments, denoted \(\sigma_i^*, \sigma_j^* \text{ and } \sigma_k^*\). Equation (8) gives producer \(i\)'s optimal assortment. See the appendix. Producers \(j\) and \(k\) have symmetric equations.

\[
\sigma_i^* = \left\{ \frac{\alpha_i c_i \left( \beta_{ij} c_i \right)^{\beta_{ij}} \left( \beta_{kk} c_k \right)^{\beta_{kk}} \omega_{ii}} {\left( \beta_{ii} - 1 \right) \left( \beta_{ij} - 1 \right) \gamma_i h_i} \right\}^{1/(\gamma_i - 1)} \quad (8)
\]
Equation (8) shows the relationship between the product line assortment and numerous factors. For example, equation (8) shows that a producer’s assortment should decrease, as the costs of producing that assortment (i.e., \( h_i \)) increase.

We now ask and answer three questions, in order of increasing difficulty. First, which factors decrease producer \( i \)'s optimal assortment \( (\sigma^*_i) \)? Second, which factors decrease producer \( i \)'s assortment relative to producer \( j \)\( (\sigma^*_i / \sigma^*_j) \)? And finally, which factors cause \( \sigma^*_i < \sigma^*_j \)?

### Which Factors Decrease Optimal Assortment?

To determine which factors decrease optimal assortments, we compute the partial derivative of \( \sigma^*_i \) with respect to each factor. Table 3 shows the signs of these derivatives.

For example, to determine the affect of own unit variable costs on assortment, we take the derivative of \( \sigma^*_i \) in equation (8), with respect to \( c_i \), and obtain:

\[
\frac{\partial \sigma^*_i}{\partial c_i} = -\sigma^*_i (\beta_i - 1) / \{ c_i (\gamma_i - 1) \}
\]

Equation (9) reveals that \( \partial \sigma^*_i / \partial c_i < 0 \). Hence, as the super-premium producer’s unit variable costs increase, the optimal super-premium assortment decreases. Although this result appears to be cost-driven, Figure 1 shows that the underlying intuition is demand-driven. As the super-premium producer’s unit variable costs increase, the super-premium producer must raise its price to cover the additional cost. As the super-premium price increases, the absolute market potential for the super-premium line decreases. As the market potential decreases, the super-premium producer’s marginal profitability of offering additional assortment decreases. With a smaller market, there is less profit in offering a greater assortment. Higher unit variable production costs, therefore, indirectly imply a smaller absolute market potential and a smaller optimal super-premium assortment.

We see from Figure 1 that any factor, which directly or indirectly decreases the super-premium producer’s market potential, can cause a decrease in the optimal super-premium assortment \( (\sigma^*_i) \). This intuition explains why the optimal super-premium assortment decreases as factors related to market size (e.g., \( \alpha_i \) or \( \omega_{ij} \)) decrease.

Decreased super-premium market potential can also come from competitive actions.

(j or k). A competitive price decrease causes a sales shift from the super-premium brand to the competitor. Hence, any factor causing competitive price decreases can also cause a decrease in the optimal super-premium assortment. For example, decreases in a competitor's unit variable costs ($c_j$ or $c_k$) or increases in a competitor's own price elasticities ($\beta_{ij}$ or $\beta_{ik}$) lead to competitive price decreases and decreased optimal super-premium assortment ($\sigma^*$).

We might suspect that if increases in competitive price elasticities ($\beta_{ij}$ or $\beta_{ik}$) cause decreases in the optimal super-premium assortment ($\sigma^*$), then increases in the super-premium price elasticity ($\beta_{ii}$) would cause increases in the optimal super-premium assortment ($\sigma^*$). However, the opposite case can occur whenever $\partial \sigma^*/\partial \beta_{ii} < 0$ in equation (10).

$$\partial \sigma^*/\partial \beta_{ii} = -\sigma^* \log \left( \beta_{ii}/(\beta_{ii} - 1) \right)/(\gamma_i - 1).$$

Of course when $\beta_{ii}$ increases, the optimal super-premium producer price decreases and that lower price increases super-premium market potential. But that lower price also implies a lower profit margin for the super-premium producer. The reduced profit margin can lessen the marginal profitability of additional assortment. If costs are sufficiently large, $c_i > 1 - (1/\beta_{ii})$, the reduced profit margin overwhelms the effect of the increased market potential. If costs are sufficiently small, $c_i < 1 - (1/\beta_{ii})$, the impact of the increased market potential dominates the impact of the reduced profit margin. See Figure 2.

Figure 2 illustrates how profit margin and market potential are opposing forces on the super-premium producer's optimal assortment. These forces were also present in equation (9). When the super-premium producer raises its price to cover additional variable costs, the super-premium producer’s profit margin increases. The increased profit margin encourages additional assortment. However, when only $c_i$ increases, the impact of decreased market potential always dominates the impact of the increased profit margin.

Costs also determine the impact of the cross price elasticity ($\beta_{ij}$). Here, $\partial \sigma^*/\partial \beta_{ij} < 0$ only when $c_i < 1 - (1/\beta_{ij})$. Larger cross price elasticities of demand imply more product substitutability. But more substitutability is good for the super-premium brand when competitive prices ($p_j$) are high. Here, the super-premium ice cream attracts buyers of lesser quality ice creams. More substitutability is bad for the super-premium brand when competitive prices are low. Here, the super-premium ice cream loses buyers to lesser quality ice creams. Competitive costs ($c_j$) help determine competitive prices. Hence, when competitive costs are sufficiently high, increased substitutability has a positive impact on super-premium demand and a positive impact on super-premium assortment. When competitive costs are sufficiently low, increased substitutability has a negative impact on super-premium demand and a negative impact on super-premium assortment.
A Note on the Linear Demand Function

Equation (11) depicts producer $i$'s profit function given a linear demand function. This function is less tractable here than our constant elasticity demand function and we cannot obtain duopoly or triopoly results. However, we can duplicate many of the results of the prior section for a monopoly.

$$\left( p_i - c_i \right) \left( e_i s_i - b_i p_i + a_i \right) - h s_i^*$$

where $e_i$, $b_i$, $a_i$, and $n$ are positive constants.

Taking the derivatives of equation (11) with respect to price ($p_i$) and assortment ($s_i$), and equating the derivatives to zero yields the optimal assortment ($s_i^*$) in equation (12).

$$s_i^* = \left( \frac{b_i h_i s_i^{n-1} + b_i a_i e_i - a_i e_i}{e_i^2} \right)$$

Taking the derivative of $s_i^*$ with respect to $c_i$ reveals how optimal assortment varies with changes in unit costs. E.g., when $n = 3$, $s_i^* = \$$e_i^* \left( -e_i \left( 24b_i^2 c_i - 24a_i b_i \right) h_i - e_i^2 \right)^{1/2} + e_i^2) / (12b_i h_i)$ and

$$\frac{\partial s_i^*}{\partial c_i} = -b_i e_i \left[ e_i^* - \left( e_i h_i \left( 24b_i^2 c_i - 24a_i b_i \right) \right) \right]^{1/2}.$$ (13)

Equation (13) shows $\frac{\partial s_i^*}{\partial c_i} < 0$. Thus, increases in the super-premium unit costs ($c_i$) decreases the optimal super-premium assortment ($s_i^* = \sigma_i^*$). This is the same result obtained for the constant elasticity demand function.

Which Factors Decrease Relative Assortment?

We have shown when a super-premium producer should decrease its assortment. Now we examine a more difficult question: "When should the super-premium producer decrease its assortment relative to a competitor's assortment?" This question is more difficult to answer because changes in most factors change both the optimal super-premium ($\sigma_i^*$) and the optimal competitor assortment ($\sigma_j^*$) in the same direction. Here, the triopoly aspect of the market is important. Unlike absolute assortment, the relative assortment between firms $i$ and $j$ depends on firm $k$'s decisions.
Equation (14) provides the relative optimal assortments between producers $i$ and $j$.

\[
\frac{\sigma_j^*}{\sigma_i^*} = \frac{\alpha_i c_i \omega_i \left( \frac{\beta_{ij} c_i}{\beta_{ij} - 1} \right)^{\gamma_j} \left( \frac{\beta_{kk} c_k}{\beta_{kk} - 1} \right)^{\beta_k} (1/(\gamma_j - 1))}{\alpha_j c_j \omega_j \left( \frac{\beta_{ji} c_i}{\beta_{ji} - 1} \right)^{\gamma_i} \left( \frac{\beta_{jj} c_j}{\beta_{jj} - 1} \right)^{\beta_j} (1/(\gamma_j - 1))},
\]

(14)

Although the mathematics are more complex for relative assortment than for absolute assortment, generally our results persist. In other words, the derivatives of equation (14) are often the same sign as the corresponding derivatives of equation (8). For example, equation (15) parallels equation (9). Equation (15) provides the derivative of equation (14), i.e., relative assortment, with respect to the super-premium producer’s variable costs ($c_i$).

\[
\frac{\partial (\sigma_j^*/\sigma_i^*)}{\partial c_i} = -\left\{ (\beta_{ii} - 1)(\gamma_j - 1) + \beta_{ii}(\gamma_j - 1) \right\} \sigma_j^*/\left\{ c_i(\gamma_j - 1)(\gamma_j - 1)\sigma_i^* \right\}.
\]

(15)

Equation (15) reveals that $\frac{\partial (\sigma_j^*/\sigma_i^*)}{\partial c_i} < 0$. Hence, both the optimal absolute and relative super-premium assortments decline as the super-premium producer’s variable cost increases. As this variable cost increases, the super-premium producer must charge a higher price. The higher price decreases the market potential for the super-premium ice cream. The decrease in the market potential overwhelms the additional revenues caused by the higher price. Moreover, the decrease in market potential is amplified by the additional assortment offered by competitor $j$. Therefore, the marginal profitability of offering additional assortment decreases and the super-premium producer should offer less relative assortment ($\sigma_j^*/\sigma_i^*$). See Figure 3.

Relative assortment also decreases with decreases in super-premium market size ($\alpha_i$), decreases in assortment importance ($\omega_i$), increases in the cost of assortment ($h_i$) and decreases in competitor $j$’s costs ($c_j$). When competitor $j$’s costs increase, competitor $j$ must charge more for its product. When competitor $j$ charges more, sales shift to the super-premium producer. The sales shift increases the marginal profitability of assortment for the super-premium producer and decreases the marginal profitability of assortment for producer $j$. Usually, we find the same intuition found for absolute assortment. However, there exists five situations when the optimal super-premium absolute assortment can decrease while the optimal relative assortment increases. We now consider each situation.

Competitor k's Unit Variable Cost ($c_k$) Decreases

Equation (16) implies that $\frac{\partial (\sigma_j^*/\sigma_k^*)}{\partial c_k}$ can be negative while $\frac{\partial \sigma_j^*/\partial c_k} > 0$.

$$\frac{\partial (\sigma_j^*/\sigma_k^*)}{\partial c_k} = \frac{[\beta_{jk}(\gamma_j - 1) - \beta_{jk}(\gamma_j - 1)] \sigma_j^*}{\{c_k(\gamma_j - 1)(\gamma_j - 1)\sigma_k^*\}}.$$  (16)

This occurs when $\beta_{jk}(\gamma_j - 1) > \beta_{jk}(\gamma_j - 1)$. Hence, when the economy producer’s costs ($c_k$) decrease, its optimal price ($p_k$) decreases, attracting some buyers from both the regular and super-premium brands. However, the regular producer suffers a greater lost market potential because $\beta_{jk} > \beta_{jk}$. The greater lost market potential causes the regular brand to decrease its assortment more than the super-premium brand.

Competitor k’s Own Price Elasticity ($\beta_{kk}$) Increases

Equation (17) implies that $\frac{\partial (\sigma_j^*/\sigma_k^*)}{\partial \beta_{kk}}$ can be positive while $\frac{\partial \sigma_j^*/\partial \beta_{kk}} < 0$.

$$\frac{\partial (\sigma_j^*/\sigma_k^*)}{\partial \beta_{kk}} = \frac{[\beta_{jk}(\gamma_j - 1) - \beta_{jk}(\gamma_j - 1)] \sigma_j^*}{\{\beta_{jk}(\gamma_j - 1)(\gamma_j - 1)\sigma_k^*\}}.$$  (17)

This occurs when $\beta_{jk}(\gamma_j - 1) > \beta_{jk}(\gamma_j - 1)$. When the economy producer’s own price elasticity ($\beta_{kk}$) increases, its price ($p_k$) decreases and it attracts buyers from both the regular and super-premium brands. However, the regular producer suffers a greater decreased market potential because $\beta_{jk} > \beta_{jk}$. The greater decreased market potential causes the regular brand to decrease its assortment more than the super-premium brand.

Competitor j’s Own Price Elasticity ($\beta_{jj}$) Increases

Equation (18) implies that $\frac{\partial (\sigma_j^*/\sigma_j^*)}{\partial \beta_{jj}}$ can be positive while $\frac{\partial \sigma_j^*/\partial \beta_{jj}} < 0$.

$$\frac{\partial (\sigma_j^*/\sigma_j^*)}{\partial \beta_{jj}} = \left\{-\frac{\beta_{jj}(\beta_{jj} - 1)(\gamma_j - 1)\log\left(\frac{\beta_{jj}c_j}{\beta_{jj} - 1}\right) - [\beta_{jj}(\gamma_j - 1)]}{(\beta_{jj} - 1)\beta_{jj}(\gamma_j - 1)(\gamma_j - 1)\sigma_j^*/\sigma_j^*}\right\}.$$  (18)

This occurs when $c_j$ is sufficiently large. Increases in competitor j’s own price elasticity ($\beta_{jj}$) cause a lower competitive price ($p_j$) which attracts buyers from the super-premium brand. The decreased potential market lowers optimal super-premium assortment. However when $c_j$ is sufficiently large, the decrease in producer j’s marginal profitability of assortment causes producer j to decrease its assortment more than the super-premium producer.

The Super-Premium Producer’s Own Price Elasticity ($\beta_{ii}$) Increases

Equation (19) implies that $\frac{\partial (\sigma_i^*/\sigma_i^*)}{\partial \beta_{ii}}$ can be positive while $\frac{\partial \sigma_i^*/\partial \beta_{ii}} < 0$.

$$\frac{\partial (\sigma_i^*/\sigma_i^*)}{\partial \beta_{ii}} = \left\{-\frac{\beta_{ii}(\beta_{ii} - 1)(\gamma_i - 1)\log\left(\frac{\beta_{ii}c_i}{\beta_{ii} - 1}\right) + [\beta_{ii}(\gamma_i - 1)]}{(\beta_{ii} - 1)\beta_{ii}(\gamma_i - 1)(\gamma_i - 1)\sigma_i^*/\sigma_i^*}\right\}.$$  (19)

This occurs when $c_i > 1 - (1/\beta_{ii})$ and $\beta_{ii}$ is sufficiently large. An increase in the super-premium producer’s own price elasticity ($\beta_{ii}$) causes the optimal super-premium price to decrease. Although the lower price increases market potential, when unit costs ($c_i$) are high, the marginal profitability of assortment decreases and optimal assortment decreases. However, when $\beta_{ii}$ is sufficiently large, producer j’s optimal price must also decrease. Facing both a decreased profit margin and a decreased potential market, producer j’s optimal assortment decreases even more than the super-premium assortment. Note that
this effect does not occur when \( c_i < 1 - \frac{1}{\beta_j} \) because, with sufficiently low costs, the super-premium producer always increases both its absolute and relative assortment.

**The Cross Price Elasticity of Demand** \( (\beta_j = \beta_i) \) **Decreases**

Equation (20) implies that \( \frac{\partial (\sigma_j^* / \sigma^*)}{\partial \beta_j} \) can be negative while \( \partial \sigma_j^* / \partial \beta_j > 0 \).

\[
\frac{\partial (\sigma_j^* / \sigma^*)}{\partial \beta_j} = \left\{ \frac{\left( \gamma_j - 1 \right) \log \left( \frac{\beta_{ij} c_i}{\beta_{jj} - 1} \right) - \log \left( \frac{\beta_{ij} c_i}{\beta_{ii} - 1} \right) (\gamma_i - 1)}{(\gamma_j - 1)(\gamma_i - 1) \sigma_j^* / \sigma^*} \right\}.
\]

(20)

This occurs when \( c_i > 1 - \frac{1}{\beta_j} \) and \( c_i \) is sufficiently large. When costs \( (c_i \) and \( c_j \)) are high, decreases in substitutability \( (\beta_j = \beta_i) \) cause prices and profit margins to increase. Consequently, assortments rise. When the super-premium producer's cost \( (c_i) \) is sufficiently large, its cost disadvantage causes a loss of buyers to competitor \( j \). Consequently, producer \( j \)'s increased market potential causes a decline in relative assortment.

**Should Häagen-Dazs Offer Fewer Flavors?**

We have examined factors which influence the absolute and relative assortments of the super-premium producer. We now seek to determine when the super-premium producer’s assortment should be less than the assortment of other producers. As an example, we ask whether Häagen-Dazs should offer less than 31 flavors. We begin by identifying the characteristics of a super-premium producer.

In nearly all cases, super-premium ice cream lines have more expensive base ingredients than economy ice cream (Robbins and Wolff 1985). An economy ice cream may have 10% butterfat while Häagen-Dazs, a super-premium ice cream, has 18% butterfat. Super-premium ice cream often has more egg solids for “French Ice Cream”, whole milk, real fruit and real nuts. Economy ice creams often use dry milk solids, cheese whey, dried cream and other less expensive ingredients (Goldbeck and Goldbeck 1976). Super-premium ice creams use more expensive sweeteners such as real sugar or honey rather than less expensive hydrolyzed starch. Super-premium ice cream often have more expensive flavorings (Robbins and Wolff 1985) such as puréed vanilla beans, vanilla extract or pure chocolate liquor rather than less expensive synthetic flavoring such as vanillan. Super-premium ice creams also provide more expensive packaging (e.g., plastic), few substitutes (e.g., fewer coloring agents) and less overrun (i.e., low air content). Overrun refers to the amount of air in an ice cream expressed as a percentage of the amount of original liquid. For example, eight gallons of liquid makes ten gallons of ice cream with a 25% overrun. High overrun indicates that an ice cream producer has whipped excess air (the cheapest ingredient) into the ice cream and, thereby, increased total volume. Häagen-Dazs’ super-premium ice cream uses 25% overrun while Baskin-Robbins 31 Flavors uses 100% overrun (Kochak 1985). These factors make super-premium ice creams more expensive to produce.

Therefore, a super-premium ice cream has a greater variable cost, \( c_i \), than other ice creams. Again, let producer \( i \) be a super-premium producer, producer \( j \) be a regular producer and producer \( k \) be an economy producer. Then \( c_i > c_j > c_k \). Setting other factors constant \( (i.e., \beta_{ir} = \beta_{ji} \) for \( r \neq i \); \( \beta_{jr} = \beta_{ji} \) for \( r \in \{ j, k \} \); \( \gamma_r = \gamma_i \) for \( r \in \{ j, k \} \); \( \alpha_r = \alpha_i \) for \( r \in \{ j, k \} \); \( \omega_r = \omega_j \) for \( r \in \{ j, k \} \); and \( h_r = h_i \) for \( r \in \{ j, k \} \) \) yields equation (21).

\[
\frac{\sigma_j^*}{\sigma_i^*} = \left( \frac{c_i}{c_j} \right)^{((\beta_{jr} + \beta_{ji} - 1)(\gamma_j - 1))} < 1 \quad \text{when} \quad c_i > c_j.
\]

(21)

*There are three categories of flavorings. Category I is totally natural. Category II is over half-natural. Category III is less than half-natural.*
Equation (21) expresses the super-premium producer's assortment as a fraction of the assortment of the regular producer, while holding demand factors and assortment costs constant. Equation (21) reveals that the super-premium producer should offer a smaller assortment than the regular producer because the super-premium producer has higher variable costs. Moreover, equation (21) reveals that the super-premium producer's relative assortment decreases as own and cross price elasticities of demand increase. Hence, as consumers become more price-sensitive or as the lines become more substitutable, the super-premium producer's relative assortment decreases even farther.

This explanation for why super-premium producers, such as Häagen-Dazs, offer a smaller assortment is, again, demand-driven. The additional variable costs of the super-premium producer merely force the price of the super-premium ice cream higher than the price of lesser quality ice creams. As a result of the higher price, the super-premium ice cream line receives fewer sales than the lesser quality ice cream. And as sales decrease, the marginal profitability of additional assortment decreases. Hence, it is the smaller market, other factors equal, of the super-premium ice cream line which causes the super-premium line to have a more limited assortment. However, the term "other factors equal" represents a strong assumption. It is unlikely that the only difference between a super-premium producer and a lesser quality producer is variable costs. Other differences may exist. The following proposition provides sufficient conditions when \( \sigma_i^* / \sigma_j^* < 1 \).

**Proposition.** The super-premium ice cream producer should have less assortment than the regular producer when the following conditions hold.

1. The super-premium producer has a greater variable production cost than lower quality producers (i.e., \( c_i > c_j > c_k \)).
2. The buyers of higher quality ice cream are no less price sensitive than the buyers of lower quality ice cream (i.e., \( \beta_i \geq \beta_j \geq 1.66 \)).
3. The market for super-premium ice cream is no larger than the market for lower quality ice cream (i.e., \( a_{ij} \leq a_{kj} \)).
4. The cost of creating assortment is no smaller for higher quality ice cream lines than lower quality ice cream lines (i.e., \( \alpha_i \omega_i = \alpha_j \omega_j \)).
5. Competition between the super-premium producer and the economy producer is no greater than competition between the regular producer and the economy producer; and the costs of the economy producer are sufficiently large (i.e., \( \beta_{ik} \leq \beta_{jk} \) and \( c_k \geq 1 - (1 / \beta_{kk}) \)). Note that the cost condition is not required when \( \beta_{ik} = \beta_{jk} \).
6. Slutsky symmetry holds (i.e., \( \beta_i = \beta_j \)).

Given the conditions in this proposition, it is not surprising that super-premium producers often offer less assortment than other producers. Most of these conditions are consistent with empirical observations. For example, usually \( h_i \geq h_j \) because of increased storage and handling costs (Robbins and Wolff 1985). Only condition 2 is likely to be violated. There are many situations when \( \beta_i < \beta_j \). For example, mark-ups for economy ice cream average 28.6%, while the mark-ups for super-premium ice creams run as high as 50%. According to our formulation, the item mark-up for a super-premium ice cream should be \( \beta_i \geq (\beta_i - 1) \). Hence, when we observe \( c_i > c_j \), empirical evidence suggests that \( \beta_i \geq (\beta_j - 1) \) which, in turn, implies that \( \beta_i < \beta_j \). We conclude that super-premium ice cream producers might have customers with a lower price sensitivity than low quality producers.

Hence, "price-sensitivity" and "relative market potential" work in opposite directions. Generally, increased variable costs, smaller absolute market potential and the presence of economy-producers all tend to decrease relative market potential and encourage less super-premium assortment. Although less price-sensitivity among super-premium cus-

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tomers could reverse the situation, competition among super-premium producers is growing and further decreasing Häagen-Dazs's potential market. Therefore, super-premium producers such as Häagen-Dazs should offer a smaller assortment than 31 flavors.

Summary and Conclusions

This paper investigated Nash equilibrium product assortments in a triopoly. Although this paper did not reveal whether Häagen-Dazs's current level of assortment is optimal, this paper does reveal several factors which would cause a super-premium producer to decrease its current assortment. Moreover, this paper does reveal several factors which would cause a super-premium producer to offer less assortment than producers of lesser quality ice creams. Those factors included: a higher variable cost for super-premium ice cream, a smaller absolute potential market for super-premium ice cream, the presence of economy ice cream producers, and, possibly, higher assortment costs for super-premium ice cream. The higher variable costs of super-premium ice cream cause that ice cream to be offered at a higher price. The higher price limits the relative market potential and a smaller relative market potential implies a lower marginal profitability for assortment. Hence, the super-premium producers should offer less assortment.

We also determined a factor which tended to encourage the super-premium ice cream producer to offer more assortment. That factor was the lower price sensitivity of the super-premium ice cream customer. Perhaps, if super-premium customers remain less price sensitive and the super-premium ice cream market continues to grow and no new super-premium producers enter the market, it will be optimal for super-premium producers to offer more assortment than nonpremium producers. In the meantime, however, we provided reasons why it is more profitable for super-premium producers to offer less assortment than nonpremium producers. Future research must determine whether our results generalize to situations when distribution is through retailers, variable costs depend on the level of assortment, elasticities vary with price levels, \( \sigma \) is interpreted as something other than assortment, and producers sell a product other than ice cream.\(^{12} \)

\(^{11}\) Provided that \( c_i \geq 1 - (1/\beta_{kk}) \).
\(^{12}\) A very early version of this paper was presented at the 1985 Joint National ORSA/TIMS Meeting held in Atlanta. The author greatly benefited from the ensuing discussion and comments. The current paper contains many ideas generated at that discussion.

Appendix

This appendix: derives equation (8), shows that equation (8) satisfies the second-order conditions for a maximum, discusses restrictions on the constants \( k_r \), and proves Proposition 1.

Derivation of Equation (8)

Taking the derivative of \( \Pi \) with respect to \( p_i \) yields:

\[
\frac{\partial \Pi}{\partial p_i} = -a_i p_i (\beta_{ii} - 1) p_i - \beta_{ii} c_i p_i^\gamma p_i^\omega_i + \sigma_i \omega_i. 
\]

Setting \( \frac{\partial \Pi}{\partial p_i} = 0 \) and solving for \( p_i^* \) reveals that \( p_i^* = (\beta_{ii} c_i)/(\beta_{ii} - 1) \). Solving \( \frac{\partial \Pi}{\partial p_r} = 0 \), where \( r \in \{ j, k \} \), reveals that \( p_r^* = (\beta_{rr} c_r)/(\beta_{rr} - 1) \).

Taking the derivative of \( \Pi \) with respect to \( \sigma_i \) yields:

\[
\frac{\partial \Pi}{\partial \sigma_i} = \frac{\sigma_i (p_i - c_i) p_i^\gamma p_i^\omega_i - \gamma_i h_i \sigma_i^{\gamma_i - 1}}{p_i^{\omega_i}}. 
\]

Substituting \( p_i^* \), \( p_r^* \) and \( p_i^* \) into the preceding equation, and solving for \( \sigma_i \) yields equation (8).

Second-Order Conditions

For a profit maximum, the second-order condition for equation (8) is that \( d^2 \Pi \) be negative definite. The matrix \( d^2 \Pi \) is negative definite when the principal minors of the Hessian are alternately negative and positive.
That is:
\[
\frac{\partial^2 \Pi}{\partial p_i^2} < 0, \\
(\frac{\partial^2 \Pi}{\partial p_i \partial p_j} \cdot \frac{\partial^2 \Pi}{\partial \sigma_i \partial \sigma_j}) - (\frac{\partial^2 \Pi}{\partial \sigma_i \partial p_i} \cdot \frac{\partial^2 \Pi}{\partial p_i \partial \sigma_i}) > 0.
\]
(24) (25)
Evaluating \( \partial^2 \Pi_i/\partial p_i^2 \), we obtain:
\[
\frac{\partial^2 \Pi_i}{\partial p_i^2} = a_i \beta_i p_i e^{-s_{-i}} \{(\beta_{ii} - 1)p_i - (\beta_{ii} + 1)c_i\} p_i^2 p_i^8 (\sigma_i \omega_i + \sigma_i \omega_i + \sigma_i \omega_i).
\]
Substituting \( p_i^* = \beta_i c_i / (\beta_{ii} - 1) \) yields:
\[
\frac{\partial^2 \Pi_i}{\partial p_i^2} \bigg|_{p_i = p_i^*} = -a_i \beta_i c_i \left( \frac{\beta_{ii} c_i}{\beta_{ii} - 1} \right) \left( \frac{\beta_{ii} + 1}{\beta_{ii} - 1} \right)^8 (\sigma_i \omega_i + \sigma_i \omega_i + \sigma_i \omega_i).
\]
Hence, \( \partial^2 \Pi_i/\partial p_i^2 < 0 \), as required. Evaluating \( \partial^2 \Pi_i/\partial \sigma_i^2 \), we obtain:
\[
\frac{\partial^2 \Pi_i}{\partial \sigma_i^2} = -(\gamma_i - 1) \gamma_i h_i e^{-s_{-i}}.
\]
Hence,
\[
\frac{\partial^2 \Pi_i}{\partial \sigma_i^2} \bigg|_{\sigma_i = \sigma_i^*} < 0.
\]
And we find that \( (\partial^3 \Pi_i/\partial p_i^2 \cdot \partial^2 \Pi_i/\partial \sigma_i^2) > 0. \)
Evaluating \( \partial^3 \Pi_i/\partial p_i^2 \), we obtain:
\[
\frac{\partial^3 \Pi_i}{\partial p_i^2 \partial \sigma_i \partial \sigma_i} = -a_i \beta_i p_i e^{-s_{-i}} \{(\beta_{ii} - 1)p_i - \beta_{ii} c_i\} p_i^2 p_i^8 (\sigma_i \omega_i).
\]
Substituting \( p_i^* = \beta_i c_i / (\beta_{ii} - 1) \) yields:
\[
\frac{\partial^3 \Pi_i}{\partial \sigma_i \partial \sigma_i} \bigg|_{p_i = p_i^*} = \left( \frac{\beta_{ii} c_i^2}{\beta_{ii}^2 - 1} \right) \left( \frac{\beta_{ii} + 1}{\beta_{ii} - 1} \right)^8 (\sigma_i \omega_i).
\]
Hence, \( (\partial^3 \Pi_i/\partial \sigma_i \partial p_i \cdot \partial^2 \Pi_i/\partial p_i \partial \sigma_i) = 0 \) and the second-order conditions are satisfied.

Restrictions on the Constants \( \kappa_v \)
Although we did not place restrictions on values for the constants \( \omega_j \), \( \gamma \), \( r \in \{i, j, k\} \), there can be implicit restrictions imposed on \( \omega_j \), by the condition that \( s_j > 0 \) and hence \( \Pi_j > 0 \). Remember that \( s_j = \omega_j \sigma_j + \omega_j \sigma_j + \omega_j \sigma_j \). Hence, whenever, \( \omega_j > 0 \) for \( r \in \{i, j, k\} \), the restriction that \( s_j > 0 \) is nonbinding, and the restriction holds for all values of \( \sigma_j \), \( \sigma_j \), and \( \sigma_j \). However, when \( \omega_j < 0 \) or \( \omega_j < 0 \) or both, then \( \omega_j \) may not be sufficiently large so that \( s_j \) and \( \Pi_j \) are positive. For example, this can occur when both \( \omega_j \) and \( \omega_j \) are negative, and both \( \sigma_j / \sigma_j \) and \( \sigma_j / \sigma_j \) are small.

Proof of Proposition
From Proposition 1, we know that \( c_i \geq 1 - (1 / \beta_{kk}) \), hence \( \beta_{kk} c_i / (\beta_{kk} - 1) \geq 1 \). We also know that \( \beta_j \leq \beta_j \). Thus, \( [\beta_{kk} c_i / (\beta_{kk} - 1)]^{s_{-k}} < 1 \). Given, \( \sigma_i \omega_j \leq \sigma_i \omega_j \), \( \gamma_j = \gamma_j \), \( h_i \geq h_i \) and \( \beta_j = \beta_j \), we need only show that the following function is less than 1.
\[
\beta_{ii} / (\beta_{ii} - 1)^{1-s_{-i}} \beta_{ii} / (\beta_{ij} - 1)^{s_{-i}} = \beta_{ii} / (\beta_{ii} - 1)^{1-s_{-i}} \beta_{ii} / (\beta_{ij} - 1)^{s_{-i}}.
\]
The preceding function is increasing in \( \beta_j \) because the derivative with respect to \( \beta_j \) is:
\[
\beta_{ii} / (\beta_{ii} - 1)^{1-s_{-i}} \beta_{ij} / (\beta_{ij} - 1)^{s_{-i}} \log \beta_{ii} / (\beta_{ij} - 1) - \log \beta_{ii} / (\beta_{ij} - 1) \beta_{ii} - 1
\]
Now, \( \beta_j < 1 \). Hence, we must prove the following function is less than or equal to 1.
\[
\beta_{ij} / (\beta_{ij} - 1)^{1-s_{-i}} \beta_{ij} / (\beta_{ij} - 1)^{s_{-i}}.
\]
Provided that \( \beta_j > 1.66 \), the preceding function is increasing in \( \beta_j \), because the derivative with respect to \( \beta_j \) is:
\[
\beta_{ii} / (\beta_{ij} - 1)^{1-s_{-i}} \beta_{ij} / (\beta_{ij} - 1)^{s_{-i}} (\beta_{ij} - 1) \beta_{ij} \log \beta_{ij} / (\beta_{ij} - 1) - 1 \beta_{ii} - 1 \beta_{ij}
\]
Now \( \beta_j \leq \beta_{ii} \), and when \( \beta_j = \beta_{ii} \) the function is 1.
References


