Equilibrium Prices in Power Exchanges with Non-convex Bids

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***Abstract*—We show uniform, linear prices in power exchange markets, such as in the Amsterdam Power Exchange (APX) Day-Ahead market or the Nord Pool Elspot market, that allow non-convex, "fill or kill" block bids by market participants may not result in an equilibrium in an economic sense, nor do they maximize surplus to market participants. We propose, as an alternative, a multi-part, discriminatory pricing mechanism that results in a market equilibrium in an economic sense and maximizes surplus for market participants. These multi-part prices do not require proceeds from outside the market to be implemented. In addition, we propose algorithms to ensure the use of linear prices for market clearing where feasible, and if not feasible, prices that minimize deviations from linear prices. We also describe a simple *pro rata* method for implementing the discriminatory multi-part prices, and discuss the degrees of freedom in pricing offered by the prices proposed through the use of simple examples.**

***Index Terms*—Auctions, Market Design, Power System Economics, Multi-part Pricing, Block Bidding.**

I. Introduction and Background

ORGANIZED power exchanges have arisen to reduce the transaction costs of trading through the determination of market prices, usually uniform, linear prices, and to maximize surplus or gains from trade that accrue to market participants (achieve economic efficiency). For uniform, linear prices to maximize surplus, given the bids submitted, they must be "equilibrium" in an economic sense. That is uniform, linear prices result in: (1) the quantities demanded and supplied are optimal for every market participant given the bids submitted, and (2) the aggregate quantity demanded is equal to the aggregate quantity supplied.

Uniform, linear prices have many attractive qualities for use in markets. Uniform, linear prices are easily computed as dual variables (shadow prices) of market optimization programs and are non-discriminatory. Decision making by market participants can be decentralized, and each participant can easily verify why its bids were accepted or rejected. This adds to the market's legitimacy and transparency. Under the assumption of convex costs, which does not hold in general for electricity production [1], uniform, linear prices are equilibrium prices in the sense described above, and economically efficient in that the surplus to market participants is maximized.

It is well understood that non-convenxities such as start-ups, minimum run levels, and minimum run times exist in electricity markets which implies linear market clearing prices may not exist and these uniform, linear prices do not maximize surplus to market participants as shown by [2], [3], [4], [5], [6], [7], and [8].

Yet economists, with the exception of [9], [10], [11], have tended to try to work around non-convexities and to de-emphasize their importance since they are inconvenient and inconsistent with many theorems in welfare economics even though, as [11] notes, they are necessary for the existence of firms. [11] and [10] summed up the dilemma caused by the inability to find linear prices: "For a theorist, the major problem presented by indivisibilities in production is the failure of the pricing test for optimality or for welfare improvements...In the presence of indivisibilities in production, [linear] prices simply don't do the jobs that they were meant to do."

In organized power exchanges such as those examined in [12], and outside of centralized unit commitment frameworks, a way to represent operational non-convexities is to allow market participants to use non-convex, block bids, which still leads to potential inefficiencies and the possibility markets will reach an equilibrium when uniform, linear prices are used. Consequently, some sort of discriminatory mechanism is required to maximize surplus and for markets to reach an equilibrium in an economic sense.

Within a centralized unit commitment framework, [3] proposes to use discriminatory linear prices, while [4], [6], and [7] propose discriminatory "disincentive" or "generalized uplift" terms that differ by market participant in order to arrive at a single uniform, linear price that differs from the shadow price on the contraint requiring supply be equal to demand. [3] does not show whether their discriminatory linear prices are equilibrium prices, and the work in [4], [6], and [7] shows their new prices are market clearing under the assumption market participants receive information on their respective "disincentive" or "generalized uplift" functions, but are silent on how market participants would receive this information in order to implement the optimal market solution in a decentralized manner.

Rather than attempting to derive linear prices to clear the market, another option is to use muti-part prices which are common in cost-of-service regulation [13] and competitive markets. [2], [5], and [8] propose using discrminatory multi-part prices to achieve a market equilibrium and economic efficiency. While [14] provides a theory of duality in integer

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programming, it has not, to our knowledge, been used to develop useful prices for markets with non-convexities. [8] provides a solution to this problem deriving multi-part, discriminatory prices that are equilibrium prices in an economic sense and showing these prices exist. They do this by putting prices not just on commodities, like electricity in a given hour, but also, when necessary, on the discrete “commodities” or variables such as the ones that control plant start ups, or in the case addressed in this paper, block bid constraints. In the face of non-convexities, market equilibrium and efficiency require such multi-part (non-linear), discriminatory prices. It is also possible to use these prices to verify that particular bids should have won or should have lost in a decentralized manner in the same manner as uniform, linear prices.

Computationally, [8] show that for any market with non-convexities that can be modeled as a mixed integer program (MIP), equilibrium-supporting prices can be found. The auctioneer takes bids from market participants and finds a solution to the MIP. The auctioneer then accepts the set of bids that maximizes bid surplus or gains from trade and calculates the equilibrium prices derived by [8] by solving a linear or concave program with the optimal solution to integer constraints inserted as equality constraints. The prices come from the dual variables including the dual variables on the added constraints.

In practice, modern commercial codes for solving MIPs can solve substantial problems in reasonable times [15] alleviating computation concerns. For example, according to [16], PJM now uses MIP to solve its day-ahead market and estimates efficiency improvements of $54 million per year. [17] simulate the loss of gains from trade from the imposition of uniform, linear prices with block bids in the Amsterdam Power Exchange (APX) and find the lost gains to be small, but find in one half of the scenarios simulated, bids are rejected that should have been accepted. This calls into question the validity of uniform, linear prices in the presence of non-convexities as equilibrium prices.

This paper shows the use of uniform, linear prices in power exchange markets that allow non-convex “fill or kill” block bids, as allowed in the Nord Pool Elspot Market [18], the Amsterdam Power Exchange (APX) [19], [20] and other European exchanges [12] neither results in a market equilibrium as defined above nor maximizes surplus. We apply the general result of [8] to block bidding and two-sided biding in power exchanges, proposing a multi-part, discriminatory, pricing mechanism that achieves a market equilibrium. We choose the power exchange environment over the centralized unit commitment context to show how the prices we derive can work to decentralize decision making where the market operator has no access to operational information from generators as is the case in the literature assuming a unit commitment environment. We also extend [8] in several ways to facilitate implementation in today’s electricity markets. First, we account explicitly for price responsive demand, which has been assumed away in literature examining non-convexities in the unit commitment context. Second, we show that prices satisfying [8] are non-confiscatory (accepted bids will never be forced to lose money) and revenue adequate (no compensation or payment from outside the market is required). Third, we propose a method to condition prices derived in the market when equilibrium prices may not be unique. The conditioning minimizes the deviations from linear prices to the extent possible while still achieving a market equilibrium. Finally, we propose a simple *pro rata* method to derive equilibrium prices that are easily implementable in markets such as Nord Pool and APX, and we discuss the degrees of freedom offered by the proposed general pricing method.

II. Power Exchange Day-Ahead Market

In this section, we describe a stylized or generalized version of a day-ahead power exchange market (PX) based on the Nord Pool Elspot Market (NPS) or the Amsterdam Power Exchange (APX). Market participants can submit bids in into the PX for the 24 hours of the following day until market closure on the day before the “dispatch day”. The PX then applies a market clearing process for every hour of the day that results in a single, linear market price and quantity for each hour. Below, we describe the characteristics of the types of bids allowed, the market clearing process used in a PX with particular attention paid to APX, and offer reasons why this process may lead to inefficient outcomes.[1]

A. Bids

A Single Bid is described by the hour t, source k, and the minimum price for sales (maximum price for purchases) in Euro/MWh, b_{kt}, of a specified quantity, Q_{kt}, in 0.1 MWh increments. An Ordinary Bid or Spot Limit Order is a set of single bids, i, of up to 25 steps in APX or 62 steps in NPS so that a spot limit order can be defined by $\{b_{kit}, Q_{kit}\}, i = 1, ...25$ in APX or $i = 1, ...62$ in NPS. Single bids and spot limit orders are flexible in that the entire bid need not be taken in the market clearing process ($q_{kit} \leq Q_{kit}$) and there is no additional constraint on accepting a bid in each hour other than that its price is less than the market price for supply bids, or greater than the market price for demand bids.

A Block Bid or Spot Block Order as it is called in APX, is a bid for which the participant offers to buy or sell the same quantity of energy, Q_{ki}, for a period of $t(i)$ consecutive hours during the trading day, at a minimum price for sales (maximum price for purchases) in b_{kit}. Spot Block Orders are inflexible and introduce a non-convexity in that they are subject to a “fill-or-kill principle”. That is, in every period for which the block bid is valid, $q_{kit} - Q_{ki} = 0, \forall t \in T$ if the bid is accepted. Additionally, the average price paid (purchase) or received (sales) by the block bid must be less than (purchase) or greater than (sales) b_{kit}. This is known as a Minimum Income Condition for sales or a Maximum Payment Condition for purchases.

[1]Information in this section as it relates to APX can be found either at www.apxgroup.com/index.php?id=35 or in [19]. We are grateful to APX for access to [19] and all the help they provided in properly characterizing their algorithm. Information relating to NPS can be found at www.nordpoolspot.com.

B. Market Clearing Process

The market-clearing process we describe is the one used in APX called Iterative Bid Matching. The process includes a "simple matching algorithm" that determines the market price and quantity in each hour over the day, P_t^* and Q_t^* under simplifying assumptions, and a check at each iteration that all block bids are feasible at the market prices. If they are not, they are eliminated. The algorithm continues until all remaining block bids are feasible. The market clearing algorithm concludes with an algorithm in an attempt, albeit imperfect as we show in the example below, to ensure that no block bids are omitted that should have been included in the dispatch. A general representation of this process is also shown in [12].

1) Simple Matching Algorithm: The algorithm determines the market price in each of the day's 24 hours. The algorithm simply sums the quantities demanded at each price for all demand bids in each hour and the quantities supplied at each price for all supply bids in each hour. It treats block bids as if they are quantity-inflexible, must-take bids. According to [12] NPS also treats block bids in the same way for the purposes of market clearing.

2) Feasibility Check for Block Bids: For each block bid, the minimum income (seller) or maximum payment (buyer) constraint is checked. This process yields a set of block bids that do not meet the minimum income/maximum payment constraint. Within that set, the block bid with the greatest loss is removed from the dispatch, and the Simple Matching Algorithm is run without it to determine a new price and quantity. This process is repeated until there are no block bids in the dispatch that lose money (fail to meet the minimum income/maximum payment constraint).

For the remaining block bids that meet the minimum income/maximum payment constraint, the quantity constraint is checked. In the set of block bids failing to meet this constraint, the one with the biggest difference between its accepted quantity and its bid quantity is dropped from the dispatch, and the Simple Matching Algorithm is run again without the removed bid to determine a new price and quantity. This process continues until each remaining block bid meets its quantity constraint.

The result of the iterative matching process is a candidate solution to the market clearing problem, with block bids, that is used as the initial solution in the "optimization algorithm" that the APX runs in an attempt to ensure that no block bids were mistakenly eliminated under Iterative Bid Matching. A full explanation of the algorithm can be found in [19]. NPS does not employ an "optimization algorithm" as used in APX according to [12].

C. Example of an APX-type Algorithm at Work

In general, schemes such as this one that involve sequential round-offs are not guaranteed to find the optimal solution of the underlying integer programming problem for optimizing surplus. Consider the following two-period example in Table I with two block bids, Bids 2 and 4. For a block bid to be accepted, it must be active in all periods for which it is valid. A negative sign signifies a bid to sell and positive bids signify bids to buy. An asterisk (*) signifies a block bid. Maximizing net surplus in the market while respecting the block bid constraints and without concern to computing prices, results in the block bid of Bid 4 being dispatched in both periods. Block Bid 2 is lower cost, but it is infeasible in Period 1. The optimal solution is shown in Table II. Now consider the APX market clearing algorithm without the optimization step alluded to above to simplify exposition. It considers block bids 2 and 4 as quantity-inflexible, must-take bids. In the first iteration, the price computed by the simple matching algorithm in period 1 is 0 (there is 225 MW of must-take block bids, 100 MW of which is needed to satisfy demand is conditionally accepted and the marginal cost to supply the next unit is 0 from the block bids assumed to be must-take). The price in period 2 is also 0 (with 225 MW of must-take block bids, 150 MW is conditionally accepted and the marginal cost of the next unit is 0 from the block bids). With these prices, a check of the block bids to satisfy the minimum income constraint shows that it is not satisfied, so the block bid with the largest losses is eliminated. Bid 4, which has offered a price of 30, is eliminated as opposed to Bid 2, which has offered a price of 5, with prices of 0.

TABLE I
TWO PERIOD BLOCK BID EXAMPLE

Bid	Period 1		Period 2	
	Price	Quantity	Price	Quantity
1	70	100		
2	-5	125*	-5	125*
3	-10	60		
4	-30	100*	-30	100*
5	-40	70	-40	160
6			90	150

TABLE II
TWO PERIOD BLOCK BID OPTIMAL, FEASIBLE SOLUTION

Bid	Period 1		Period 2	
	Price	Quantity	Price	Quantity
1	70	100		
2	-5	0	-5	0
3	-10	0		
4	-30	100*	-30	100*
5	-40	0	-40	50
6			90	150
Surplus	4000		8500	

In the second iteration, the price in Period 1 is again zero as there is more block bid offered than demand from Bid 2. In Period 2, all of the block bid is accepted and then 25 MW from Bid 5 is taken to satisfy demand. The minimum income condition for Bid 2 is satisfied as $(0-5)*125+(40-5)*125>0$. As there are no other block bids remaining, the APX algorithm checks the quantity constraint for Bid 2, but it is not satisfied for period 1 as the Bid 2 quantity of 125 MW exceeds the 100 MW demand. Bid 2 is then eliminated.

We then proceed to the third iteration. As there are no block bids remaining, it is easy to see the price in period 1 is 40 and the price in period 2 is 40. In period 1, Bid 3 is dispatched at 60 units and Bid 5 is dispatched at 40 units. In period 2, Bid 5 is dispatched for 150 units. The total surplus in the candidate solution in Table III is 12300 which is less than the surplus in the optimal solution of Table II of 12500. In the actual APX algorithm the candidate soultion would be taken to the "optimization algorithm", which we are omitting here as it is not often used in other exchanges [12]. The prices in the candidate solution shown in Table III are not equilibrium prices. At these prices both block bids, Bids 2 and 4, would wish to be dispatched at those prices, but are forced not to run and thus result in non-optimal behavior which violates one of the conditions for a market equilibrium. No set of linear prices achieve a market equilibrium given the block bids. For linear prices to be equilibrium prices for both periods require $P_1 \leq 10, P_2 = 40$, and $P_1 + P_2 \geq 60$, where P_1 and P_2 are the prices in periods 1 and 2 respectively. This set has no feasible solution. Therefore, we need multi-part pricing to achieve a market equilibrium. One way to reach a market equilibrium and maximize surplus is to make a lump-sum payment to Bid 4 of 1000 at the prices $P_1 = 10, P_2 = 40$.

TABLE III
FIRST AND SECOND ITERATIONS ELIMINATE BLOCK BIDS AND THIRD ITERATION FINDS CANDIDATE SOLUTION

Bid	Period 1		Period 2	
	Price	Quantity	Price	Quantity
1	70	100		
2				
3	-10	60		
4				
5	-40	40	-40	150
6			90	150
Price	40		40	
Surplus	4800		7500	

Next, we offer a pricing mechanism for power exchanges to maximize the efficiency of trading and, to the extent possible, develop a way to maximize the use of linear prices to achieve a market equilibrium.

III. MARKET EFFICIENCY AND PRICING

This section describes a method that maximizes the value produced by the auction by maximizing the auction surplus and deriving prices that achieve a market equilibrium. We also define a settlement process to satisfy market equilibrium conditions. The actual market mechanism may have locational aspects, but these aspects do not invalidate, and only complicate the presentation. We define the following notation:

- B is the set of offers to buy;
- S is the set of offers to sell;
- LO is the set of Limit Orders;
- BO is the set of Block Orders;
- $i \in B, S$ is the index of offers to buy or sell. We emphasize i *is not* an index of agents;
- t is the index of time;
- $t(i)$ is the time block in which a block bid is valid. $t_{start(i)}$ is the start time for the block and $t_{stop(i)}$ is the stop time for the block bid $i \in BO$;
- b_{it} is the bid price for bid i in period t; [2]
- Q_{it} is the maximum quantity that can be chosen for bid i at time t with $Q_{it} > 0$;
- For block bids Q_{it} is the same value $\forall t \in t(i)$, and thus we drop the t subscript and use Q_i to denote the block bid quantity. That is $Q_{it} = Q_i, \forall t \in t(i)$;
- q_{it} is the quantity chosen for bid i in period t;
- z_i is a binary variable for Block Orders to indicate if a block order is chosen, $z_i = 1$ if it is chosen, 0 otherwise.

We have defined bids for the purpose of our presentation as single bids as defined in Section II-A without loss of generality.[3]

Daily, after bid submission, the market operator/auctioneer chooses quantities q_{it} and block bids to activate z_i to solve the following mixed integer program (MIP) in equations (1)-(6):

$$MS = \max_{\substack{q_{it} \\ \forall i \in B,S}} \sum_t \sum_{i \in B} b_{it} q_{it} - \sum_t \sum_{i \in S} b_{it} q_{it} \tag{1}$$

subject to

$$\sum_{i \in B} q_{it} - \sum_{i \in S} q_{it} = 0, \forall t \in T = \{1, ..., 24\} \tag{2}$$

$$q_{it} \leq Q_{it}, \forall i \in LO, \forall t \in T \tag{3}$$

$$q_{it} - z_i Q_i = 0, \forall i \in BO, \quad \forall t \in t(i) = \{t_{start(i)}, ..., t_{stop(i)}\} \tag{4}$$

$$z_i \in \{0, 1\} \tag{5}$$

$$q_{it} \geq 0, \forall i, t. \tag{6}$$

Let $*$ indicate an optimal solution. If $q^*_{it} > 0$, the bid is accepted in part or totally. If $q^*_{it} = 0$, the bid is not accepted. If $z^*_i = 1$ the block bid for i is accepted, and if $z^*_i = 0$ the block bid for i is not accepted.

To calculate the prices paid for the settlement, we formulate the MIP defined in (1)-(6) as a linear program (LP) with the optimal integer solutions (z^*_i) inserted as equality constraints as suggested by [8]. This problem is defined by (7)-(12) with the only difference being the equality constraint in (11).

$$MSLP = \max_{\substack{q_{it} \\ \forall i \in B,S}} \sum_t \sum_{i \in B} b_{it} q_{it} - \sum_t \sum_{i \in S} b_{it} q_{it} \tag{7}$$

subject to

$$\sum_{i \in B} q_{it} - \sum_{i \in S} q_{it} = 0, \forall t \in T = \{1, ..., 24\} \qquad (\lambda_t) \tag{8}$$

$$q_{it} \leq Q_{it}, \forall i \in LO, \forall t \in T \qquad (\beta_{it}) \tag{9}$$

$$q_{it} - z_i Q_i = 0, \forall i \in BO,$$

[2] The bids can also be represented by piecewise linear functions as follows: Let K_t be the number of steps or breaks in the bid in period t given by $(b_{itk}, q_{itk}), k = 1,, K_t$. In the objective function: $\sum_k b_{itk} q_{itk} w_{itk}$. In the market clearing constraint: $q_{it} = \sum_k q_{itk} w_{itk}$. There is an additional constraint: $\sum_k w_{itk} = 1$. Since incorporating these equations does not change any of the important questions addressed and it complicates the notation, we will not use them in the formulation.

[3] We have chosen this presentation to economize on notation. We could have defined bids as ordinary bids as done in Section II-A, but it adds an additional subscript to variables and an additional nested summation to optimization programs, but does not change our results at all. It also follows the formulation of bids in [3].

$$\forall t \in t(i) = \{t_{start(i)}, ..., t_{stop(i)}\} \quad (\alpha_{it}) \quad (10)$$

$$z_i = z_i^* \quad (\delta_i) \quad (11)$$

$$q_{it} \geq 0, \forall i, t \quad (12)$$

Again, following [8] we define the dual to the linear program defined in (7)-(12) to compute the dual variables. Let λ_t, β_{it}, α_{it}, and δ_i be the dual variables associated with the hourly market clearing (8), individual spot limit order (9), individual block order (10), and block bid acceptance (11) constraints respectively. β_{it} and α_{it} also represent the per MWh bid surplus, or the difference between the submitted bid b_{it} and the shadow price λ_t for limit orders and block orders respectively and δ_i the total block bid surplus summed over all hours for which the block bid would be activated.

Thus, the dual problem described by equations (13)-(19) is

$$MSD = \min_{\substack{\beta_{it}, \alpha_{it}, \delta_i \\ \forall i \in B,S}} \sum_t \sum_{i \in LO} Q_{it}\beta_{it} + \sum_{i \in BO} z_i^* \delta_i \quad (13)$$

subject to

$$\lambda_t + \beta_{it} \geq b_{it}, \forall i \in LO \cap B, t \quad (14)$$

$$-\lambda_t + \beta_{it} \geq -b_{it}, \forall i \in LO \cap S, t \quad (15)$$

$$\lambda_t + \alpha_{it} = b_{it}, \forall i \in BO \cap B, t \in t(i) \quad (16)$$

$$-\lambda_t + \alpha_{it} = -b_{it}, \forall i \in BO \cap S, t \in t(i) \quad (17)$$

$$\delta_i - \sum_{t \in t(i)} Q_i \alpha_{it} = 0 \quad (18)$$

$$\beta_{it} \geq 0; \alpha_{it}, \delta_i \ unrestricted, \forall i, t \quad (19)$$

Theorem 1: Surplus Equivalence. The value of the objective function MS in (1) is equal to the value of the objective function $MSLP$ in (7) and is equal to the value of the objective function MSD in (13) which is greater than or equal to zero: $MS = MSLP = MSD \geq 0$.

Proof: Since by construction fixing variables at their optimal values and resolving does not change the optimal solution value, $MS = MSLP$. Since the objective function in (7) is the objective of a linear program, by strong duality, $MSLP = MSD$. Since $q_{it} = 0$ is a feasible solution $\forall i, t$ with $MS = 0$, any other solution must result in $MS \geq 0$. QED

A. Properties of an Optimal Solution

From here on, the value of the variables will be optimal unless otherwise noted. We describe the optimal solution for buyers' and sellers' spot orders and buyers and sellers with block orders below respectively.

For both buyers and sellers who submit spot limit orders, if a bid is accepted, $q_{it} > 0$. By complementary slackness, for buyers, $b_{it} - \lambda_t = \beta_{it}$ and for sellers, $\lambda_t - b_{it} = \beta_{it}$. Since $\beta_{it} \geq 0$, for buyers, $\lambda_t \leq b_{it}$, the market price is less than or equal to the bid (willingness to pay). In addition, since $\beta_{it} \geq 0$, for sellers, $\lambda_t \geq b_{it}$, the market price is greater than or equal to the bid (willingness to accept). For both buyers and sellers, the bid surplus or profit, $\beta_{it} q_{it} \geq 0$, is always non-negative.

For both buyers and sellers who submit spot limit orders, if $0 < q_{it} < Q_{it}$, then by complimentary slackness $\beta_{it} = 0$ and $\lambda_t = b_{it}$. The bid sets the linear price and is said to be marginal. If $q_{it} = Q_{it}$, by complementary slackness, $\beta_{it} \geq 0$ and the bid is said to be inframarginal.

For both buyers and sellers who submit spot limit orders, if a bid is not accepted, $q_{it} = 0$, and by complementary slackness, $\beta_{it} = 0$. For buyers, $\lambda_t \geq b_{it}$, the market price is greater than or equal to the bid (willingness to pay), and for sellers, $\lambda_t \leq b_{it}$, the market price is less than or equal to the bid (willingness to accept).

For buyers who submit block orders, from (16) $b_{it} - \lambda_t - \alpha_{it} = 0$ so that $\sum_{t \in t(i)} (b_{it} - \lambda_t - \alpha_{it}) Q_i = 0$. Substituting (18) we obtain $\delta_i = Q_i \sum_{t \in t(i)} (b_{it} - \lambda_t)$. δ_i is bid surplus for block bids from uniform, linear prices λ_t.

For sellers who submit block orders, from (17) $\lambda_t - b_{it} - \alpha_{it} = 0$ so that $\sum_{t \in t(i)} (\lambda_t - b_{it} - \alpha_{it}) Q_i = 0$. Substituting (18) we obtain $\delta_i = Q_i \sum_{t \in t(i)} (\lambda_t - b_{it})$. δ_i is bid surplus (profit) for block bids from uniform, linear prices λ_t.

For block bids in general, if $\delta_i \geq 0$, the bid surplus under uniform, linear prices is nonnegative. If $\delta_i < 0$, the bid surplus under uniform, linear prices is negative. If a block order is accepted, $q_{it} = Q_i, \forall t \in t(i)$ and $\delta_i \geq 0$, then linear prices are *non-confiscatory* in the sense of Definition 2 below. If a block order is accepted, $q_{it} = Q_i, \forall t \in t(i)$ and $\delta_i < 0$, then linear prices are *confiscatory* in the sense of Definition 2 below.

If a block order is not accepted, $q_{it} = 0, \forall t \in t(i)$, and $\delta_i \geq 0$, then linear prices improperly signal acceptance of the bid. This indicates that for some $t \in t(i), 0 < q_{it} < Q_i$, the bid would be accepted but the fill-or-kill requirement does not permit it. If the block order is not accepted, $q_{it} = 0, \forall t \in t(i)$ and $\delta_i \leq 0$, then linear prices properly signal rejection of the bid.

B. Conditions for Linear Equilibrium Prices

Definition 1: Linear Equilibrium Prices. The set of linear prices λ_t for $t = 1, ..., 24$ are equilibrium prices given submitted bids if for hours $t = 1, ..., 24$:

1) $\forall i \in LO \cap B, q_{it} > 0$ implies $\lambda_t \leq b_{it}$ and $q_{it} = 0$ implies $\lambda_t \geq b_{it}$;
2) $\forall i \in LO \cap S, q_{it} > 0$ implies $\lambda_t \geq b_{it}$ and $q_{it} = 0$ implies $\lambda_t \leq b_{it}$;
3) $\forall i \in BO \cap B, q_{it} > 0$ implies $\sum_{t \in t(i)} (-\lambda_t + b_{it}) \geq 0$ and $q_{it} = 0$ implies $\sum_{t \in t(i)} (-\lambda_t + b_{it}) \leq 0$;
4) $\forall i \in BO \cap S, q_{it} > 0$ implies $\sum_{t \in t(i)} (\lambda_t - b_{it}) \geq 0$ and $q_{it} = 0$ implies $\sum_{t \in t(i)} (\lambda_t - b_{it}) \leq 0$;
5) $\sum_{i \in S} q_{it} = \sum_{i \in B} q_{it}$.

Conditions 1) and 2) are optimizing conditions for individual bidders submitting Limit Orders which state that bidders must have non-negative surplus to be dispatched in the market. 3) and 4) are optimizing conditions for individual bidders submitting Block Orders stating that the block bids must gain non-negative surplus if they are to be activated in equilibrium. 5) is the market clearing condition.

Our definition says that any market bidder with positive purchases or sales, regardless of the type of bid, must have non-negative surplus under uniform, linear prices. Any market bidder with quantities of zero would necessarily have non-positive surplus under uniform, linear prices if their bids were activated. This says that market-clearing prices result in sellers being paid at least what they bid and buyers paying no more than what they bid. This leads us to the following result that shows when uniform, linear prices are sufficient to clear the market.

Theorem 2: Linear Price Equilibrium Theorem. Linear prices λ_t for hours $t = 1, ..., 24$, from an optimal solution are equilibrium prices given the submitted bids if and only if $\delta_i = \sum_{t\in t(i)} Q_i\alpha_{it} \geq 0$, when $z_i = 1, \forall i \in BO$, and $\delta_i = \sum_{t\in t(i)} Q_i\alpha_{it} \leq 0$, when $z_i = 0, \forall i \in BO$.

Proof: First, we show that if $\delta_i \geq 0$ when $z_i = 1$, and $\delta_i \leq 0$ when $z_i = 0, \forall i \in BO$, linear prices λ_t are market clearing.

Conditions 1), 2), and 5) from our definition of market clearing are satisfied in a straightforward manner from the constraints of the linear program and its dual defined in (7)-(12) and (13)-(19) respectively. We must show that 3) and 4) from our definition hold. We will first show sufficiency, then necessity.

If $\delta_i \geq 0$ and $z_i = 1$, we know $\sum_{t\in t(i)} Q_i\alpha_{it} \geq 0$, and thus $\sum_{t\in t(i)} \alpha_{it} \geq 0$. From (16) in the dual problem, $\lambda_t + \alpha_{it} = b_{it}, \forall t \in t(i)$, thus $\sum_{t\in t(i)}(\lambda_t - b_{it}) \leq 0$, satisfying the first part of condition 3) of our definition.

If $\delta_i \leq 0$ and $z_i = 0$, we know $\sum_{t\in t(i)} Q_i\alpha_{it} \leq 0$ and thus $\sum_{t\in t(i)} \alpha_{it} \leq 0$. From (16) in the dual problem, $\lambda_t + \alpha_{it} = b_{it}, \forall t \in t(i)$, thus $\sum_{t\in t(i)}(\lambda_t - b_{it}) \geq 0$, satisfying the second part of condition 3) of our definition.

Analogously for sellers, if $\delta_i \geq 0$ and $z_i = 1$, we know $\sum_{t\in t(i)} Q_i\alpha_{it} \geq 0$, and thus $\sum_{t\in t(i)} \alpha_{it} \geq 0$. From (17) in the dual problem, $-\lambda_t + \alpha_{it} = -b_{it}, \forall t \in t(i)$, thus $\sum_{t\in t(i)}(\lambda_t - b_{it}) \geq 0$, satisfying the first part of condition 4) of our definition.

If $\delta_i \leq 0$ and $z_i = 0$, we know $\sum_{t\in t(i)} Q_i\alpha_{it} \leq 0$ and thus $\sum_{t\in t(i)} \alpha_{it} \leq 0$. From (17) in the dual problem, $-\lambda_t + \alpha_{it} = -b_{it}, \forall t \in t(i)$, thus $\sum_{t\in t(i)}(\lambda_t - b_{it}) \leq 0$, satisfying the second part of condition 4) of our definition.

Now we show that if linear prices, λ_t, are market clearing, $\delta_i \geq 0$ when $z_i = 1$, and $\delta_i \leq 0$ when $z_i = 0, \forall i \in BO$.

From 3) in our definition of market clearing, $\forall i \in BO \cap B, q_{it} > 0$ implies $\sum_{t\in t(i)}(\lambda_t - b_{it}) \leq 0$, and $\forall i \in BO \cap B, q_{it} = 0$ implies $\sum_{t\in t(i)}(\lambda_t - b_{it}) \geq 0$. $\forall i \in BO \cap B, q_{it} > 0$ implies $z_i = 1$, and $q_{it} = 0$ implies $z_i = 0$.

From (16) of the dual problem, $\lambda_t + \alpha_{it} = b_{it}, \forall t \in t(i)$. Since $\sum_{t\in t(i)}(\lambda_t - b_{it}) \leq 0$ for $q_{it} > 0$, $\sum_{t\in t(i)} \alpha_{it} \geq 0$. Multiplying by Q_i gives us $\delta_i \geq 0$. Similarly, since $\sum_{t\in t(i)}(\lambda_t - b_{it}) \geq 0$ for $q_{it} = 0$, $\sum_{t\in t(i)} \alpha_{it} \leq 0$ yielding $\delta_i \leq 0$.

From 4) in our definition for market clearing, $\forall i \in BO \cap S, q_{it} > 0$ implies $\sum_{t\in t(i)}(\lambda_t - b_{it}) \geq 0$, and $\forall i \in BO \cap S, q_{it} = 0$ implies $\sum_{t\in t(i)}(\lambda_t - b_{it}) \leq 0$. $\forall i \in BO \cap S, q_{it} > 0$ implies $z_i = 1$, and $q_{it} = 0$ implies $z_i = 0$.

From (17) of the dual problem, $-\lambda_t + \alpha_{it} = -b_{it}, \forall t \in t(i)$. Since $\sum_{t\in t(i)}(\lambda_t - b_{it}) \geq 0$ for $q_{it} > 0$, $\sum_{t\in t(i)} \alpha_{it} \geq 0$. Multiplying by Q_i gives us $\delta_i \geq 0$. Similarly, since $\sum_{t\in t(i)}(\lambda_t - b_{it}) \leq 0$ for $q_{it} = 0$, $\sum_{t\in t(i)} \alpha_{it} \leq 0$ yielding $\delta_i \leq 0$. ♣

IV. Multi-Part, Discriminatory Prices as Equilibrium Prices

A. A Simple Example Revisited

In spite of knowing the optimal solution, the auctioneer cannot pay less than a bid to sell or charge more than the bid to buy. We must find a way to satisfy this non-confiscation requirement and maintain optimality. Moreover, the auctioneer must also ensure that block bids that have bids less than λ_t, but are quantity infeasible receive negative surplus. Multi-part or discriminatory pricing is required for efficient pricing to maintain optimality. Next we show an example.

Consider a market where demand is fixed (perfectly inelastic). If there were a block bid seller that was activated such that $\delta_i = \sum_{t\in t(i)} \alpha_{it}Q_i < 0$, a discriminatory price/payment for bidder i equal to $-\delta_i$ would make the overall bid surplus zero for supplier i, and would result in non-confiscatory prices. This can easily be seen in the APX two-period example shown in Tables I, II, and III of Section II. For Bid 4, the only quantity feasible block bid in that example, at energy prices of $P_1 = 10$ and $P_2 = 40$ that would exist if Bid 4 were the only block bid, Bid 4 loses 1000. A simple discriminatory payment of 1000, collected from demand, to Bid 4 in addition to the prices of 10 and 40 will result in an overall surplus of zero for Bid 4 and is non-confiscatory. For Bid 2, the issue is different. It would wish to be dispatched at prices greater than 5 but it is quantity infeasible. To ensure that Bid 2 does not self-dispatch itself in a manner different from its block bid, it could be forced to pay back the positive surplus it would gain by dispatching itself. In the example in Tables I, II, and III in Section II, a payment to the market greater than or equal to 40 for each MWh it produces would keep Bid 2 from dispatching in a manner different from its bid. Such prices also preserve the optimality of the solution as well since demand is assumed to be perfectly inelastic.

B. Defining Multi-part Prices with Price Responsive Demand in the Market

The case of inelastic demand is easy to handle because the discriminatory payment can be collected from demand in any manner without affecting efficiency. Including price responsive demand, as we do here, is more complicated. If $z_i^* = 0$, the market settlement is zero for those block bids not accepted. For $z_i^* = 1$, we split the block orders into two groups, those with positive bidding surplus and those with negative bidding surplus at the marginal cost of delivering one more unit to the market, λ_t. Define the set of positive surplus block orders as
$BO^+ = \{i : \delta_i \geq 0 \text{ and } z_i^* = 1\}$
and the set of negative surplus block orders as
$BO^- = \{i : \delta_i < 0 \text{ and } z_i^* = 1\}$.

We define $\beta'_{it}, \alpha'_{it}$, and δ'_i as redistribution variables or prices for the settlement. The inclusion of $\beta'_{it}, \alpha'_{it}$, and δ'_i in the settlement along with λ_t creates a discriminatory pricing scheme. The redistribution variables, or prices, must be non-confiscatory and revenue neutral as defined below.

Definition 2: Non-confiscation. Non-confiscation requires that a seller, if its bid is accepted, will receive at a minimum its bid cost. It requires that a buyer, if its bid is accepted, pay no more than its bid. Mathematically, this requires

$$q_{it}\beta_{it} \geq q_{it}\beta'_{it} \geq 0, i \in LO; \quad (20)$$

$$\delta_i = \sum_{t\in t(i)} Q_i\alpha_{it} \geq \delta'_i = \sum_{t\in t(i)} Q_i\alpha'_{it} \geq 0, i \in BO^+; \quad (21)$$

$$0 > \delta_i = \sum_{t\in t(i)} Q_i\alpha_{it} \geq \delta'_i = \sum_{t\in t(i)} Q_i\alpha'_{it}, i \in BO^-; \quad (22)$$

$\delta'_i > 0$ is a discriminatory (or non-linear) payment to the market, and $\delta'_i < 0$ is a discriminatory (or non-linear) payment received from the market.

Definition 3: Revenue Adequacy. Revenue adequacy requires the amounts collected from bids equal or exceed the payments made to bids.

$$\sum_{i\in LO}\sum_{t\in t(i)} q_{it}\beta'_{it} + \sum_{i\in BO^+}\sum_{t\in t(i)} z^*_i Q_i\alpha'_{it} + \sum_{i\in BO^-}\sum_{t\in t(i)} z^*_i Q_i\alpha'_{it} = \sum_{i\in LO}\sum_{t\in t(i)} q_{it}\beta'_{it} + \sum_{i\in BO^+} z^*_i\delta'_i + \sum_{i\in BO^-} z^*_i\delta'_i \geq 0. \quad (23)$$

Definition 4: Market Settlement. Each limit order bid i at time t is settled by payment s_{it}. Each block order bid i valid $\forall t \in t(i)$ is settled by payment $s_{i,t(i)}$. If s_{it} or $s_{i,t(i)} < 0$, the settlement is a payment to bid i. If s_{it} or $s_{i,t(i)} > 0$, the settlement is a payment from bid i.

A discriminatory and revenue neutral market settlement for limit orders at time t is, $\forall i \in LO$ and $q_{it} > 0$,

1) $s_{it}(\lambda_t, \beta'_{it}, q_{it}) = q_{it}(\lambda_t + \beta'_{it}), i \in LO \cap B$;
2) $s_{it}(\lambda_t, \beta'_{it}, q_{it}) = -q_{it}(\lambda_t - \beta'_{it}), i \in LO \cap S$.

A discriminatory and revenue neutral market settlement for block orders is $\forall i \in BO$ and $\forall t \in t(i)$,

3) $s_{i,t(i)}(\lambda_t, \alpha'_{it}, q_{it}) = z^*_i \sum_{t\in t(i)}(\lambda_t + \alpha'_{it})Q_i$;
$s_{i,t(i)}(\lambda_t, \delta'_i, q_{it}) = z^*_i(\sum_{t\in t(i)}(\lambda_t Q_i) + \delta'_i), i \in BO \cap B$;
4) $s_{i,t(i)}(\lambda_t, \alpha'_{it}, q_{it}) = -z^*_i \sum_{t\in t(i)}(\lambda_t - \alpha'_{it})Q_{it}$;
$s_{i,t(i)}(\lambda_t, \delta'_i, q_{it}) = -z^*_i(\sum_{t\in t(i)}(\lambda_t Q_i) - \delta'_i), i \in BO \cap S$.

This settlement can be characterized as a two-part settlement consisting of a linear price (λ_t) that is independent if bid i, a discriminatory, non-linear, lump-sum payment, $q_{it}\beta'_{it}$, for limit orders, and a discriminatory, non-linear, lump-sum payment, $z^*_i \sum_{t\in t(i)} \alpha'_{it}Q_i = z^*_i\delta'_i$, for block orders that are specific to bid i.

Definition 5: Equilibrium Multi-part Prices. The set of linear prices λ_t for $t = 1, ..., 24$ and non-anonymous market payments β'_{it} and δ'_i are equilibrium prices given submitted bids if for hours $t = 1, ..., 24$:

1) Buyers and sellers who submit limit orders and have their bids accepted, $\forall i \in LO$, $q_{it} > 0$ implies $(\lambda_t + \beta'_{it}) \leq b_{it}$, i.e. the multi-part price is less than or equal to the bid for buyers; and $(\lambda_t - \beta'_{it}) \geq b_{it}$, i.e. the multi-part price is greater than or equal to the bid for sellers;
2) Buyers and sellers who submit limit orders and have their bids rejected, $\forall i \in LO$, $q_{it} = 0$ implies $\beta'_{it} = 0$, $\lambda_t \geq b_{it}$, i.e. the multi-part market price is greater than or equal to the bid for buyers, and $\lambda_t \leq b_{it}$, i.e. the multi-part market price is less than or equal to the bid for sellers;
3) For buyers and sellers who submit block orders and have their bids accepted, $\forall i \in BO$, $q_{it} = Q_i \forall t \in t(i)$ implies for buyers the sum of the multi-part prices over all periods the block bid is in force is less than or equal to the sum of the bid over all periods the block bid is in force, i.e. $\sum_{t\in t(i)}(\lambda_t + \alpha'_{it}) \leq \sum_{t\in t(i)} b_{it}$, and for sellers the sum of the multi-part prices over all periods the block bid is in force is greater than or equal to the sum of the bid over all periods the block bid is in force, i.e. $\sum_{t\in t(i)}(\lambda_t + \alpha'_{it}) \geq \sum_{t\in t(i)} b_{it}$;
4) For buyers and sellers who submit block orders and have their bids rejected, $\forall i \in BO$, $q_{it} = 0 \forall t \in t(i)$ implies for buyers the sum of the multi-part prices over all periods the block bid is in force is greater than or equal to the sum of the bid over all periods the block bid is in force, i.e. $\sum_{t\in t(i)}(\lambda_t + \alpha'_{it}) \geq \sum_{t\in t(i)} b_{it}$, and for sellers the sum of the multi-part prices over all periods the block bid is in force is less than or equal to the sum of the bid over all periods the block bid is in force, i.e. $\sum_{t\in t(i)}(\lambda_t + \alpha'_{it}) \leq \sum_{t\in t(i)} b_{it}$;
5) $\sum_{i\in S} q_{it} = \sum_{i\in B} q_{it}$.

Conditions 1) and 2) are optimizing conditions for individual limit orders while ensuring the redistribution is non-confiscatory. Condition 1) assures accepted individual limit orders will have non-negative surplus, and Condition 2) assures rejected limit orders see prices that indicate why they have been rejected, how bids would need to be changed to be accepted, and that no surplus redistribution takes place. Conditions 3) and 4) are optimizing conditions for block orders and assures that the redistribution is non-confiscatory. Condition 3) assures accepted block orders obtain non-negative surplus, while Condition 4) assures rejected block orders see why they were rejected with the multi-part price informing the bidder of the minimum amount that the bid would need to be changed to be accepted, and that no surplus re-distribution takes place. Condition 5) balances supply and demand.

Theorem 3: Multi-part Price Equilibrium Theorem There exists a set of revenue adequate, non-confiscatory, multi-part prices for limit orders (λ_t, β'_{it}) and block orders $(\lambda_t, \alpha'_{it})$ or (λ_t, δ'_i), for all i and t that can be constructed from the optimal dual problem defined in (13)-(19) such that they form an equilibrium as defined in Definition 5 that maximizes gains from trade.

Proof: See Appendix.

V. Conditioning Method for Finding Linear Prices or Minimizing Deviations from Linear Prices

We start with the assumption that it is desirable to transfer as much of the surplus as possible through linear prices while maintaining optimality. We also note the market surplus maximization problem may not have a unique, non-degenerate solution, nor will the dual problem. Consequently, prices may not be unique. From the dual problem defined by equations

(13)-(19), we obtain the surplus values using only linear prices over all hours for which a block bid is in effect, δ_i, for buyers and sellers respectively:

$$\delta_i - \sum_{t\in t(i)} Q_i(\lambda_t - b_{it}) = 0, \forall i \in BO \cap B \tag{24}$$

$$\delta_i - \sum_{t\in t(i)} Q_i(-\lambda_t + b_{it}) = 0, \forall i \in BO \cap S. \tag{25}$$

We can condition the dual solution to find linear prices, if they exist, by adding the following constraints to the dual problem in (13)-(19):

$$\delta_i \geq 0, \forall z_i^* = 1 \tag{26}$$

$$\delta_i \leq 0, \forall z_i^* = 0. \tag{27}$$

If a feasible solution exists to the conditioned dual problem defined by (13)-(19) and (26)-(27), then the entire settlement can conducted using linear prices. However, if no solution exists, as is quite possible with block bids, we can condition the dual solution to choose prices that maximize the transfer of surplus through linear prices, while recognizing non-linear prices are necessary to clear the market.

We now define y^+ and y^- to be the sum of surplus values on accepted and non-accepted block bids respectively. The solution to the following problem maximizes the transfer of surplus in the entire market through linear prices, or stated differently, minimizes the sum of discriminatory payments made in the market.

$$\max_{y^+, y^-} y^+ - y^- \tag{28}$$

subject to

$$y^+ - \sum_{i:z_i^*=1} \delta_i \leq 0, \tag{29}$$

$$\sum_{i:z_i^*=0} \delta_i - y^- \leq 0, \tag{30}$$

$$MSD = \sum_t \sum_{i\in LO} Q_{it}\beta_{it} + \sum_{i\in BO} z_i^*\delta_i \tag{31}$$

$$\lambda_t + \beta_{it} \geq b_{it}, \forall i \in LO \cap B, t \tag{32}$$

$$-\lambda_t + \beta_{it} \geq -b_{it}, \forall i \in LO \cap S, t \tag{33}$$

$$\lambda_t + \alpha_{it} = b_{it}, \forall i \in BO \cap B, t \in t(i) \tag{34}$$

$$-\lambda_t + \alpha_{it} = -b_{it}, \forall i \in BO \cap S, t \in t(i) \tag{35}$$

$$\delta_i - \sum_{t\in t(i)} Q_i\alpha_{it} = 0 \tag{36}$$

$$\beta_{it} \geq 0; \alpha_{it}, \delta_i \ unrestricted, \forall i, t \tag{37}$$

(31)-(37) ensure a dual optimal solution from (13)-(19). Moreover, this problem differs from the problem defined by (13)-(19) and (26)-(27) in that a solution will always exist and the problem will result in linear prices when linear prices clear the market.

An alternative formulation to (28)-(37) is to minimize the largest discriminatory payment rather than minimizing the total discriminatory payments made in the market. This formulation only requires changing the constraints in (29) and (30) to, respectively

$$y^+ - \delta_i \leq 0; \tag{38}$$

$$\delta_i - y^- \leq 0. \tag{39}$$

VI. Practical Matters in Computing Multi-part Prices to Clear the Market

Theorem 3 does not provide a prescription for redistributing the surplus to clear the market, but simply states it is possible to do so while maintaining non-confiscation and revenue neutrality. The algorithm for conditioning prices to minimize the deviations from linear prices provides some direction, but does not provide a full prescription for redistributing surplus to derive market clearing prices. Consequently, the prices and settlement rules we have defined leave degrees of freedom to account for local market culture and customs.

A. A Simple Rule for Implementing Market Clearing Prices

We offer a simple *pro rata* market rule. Define the "uplift" needed to make settlements to block bid non-confiscatory individually and in aggregate by (40), and define the surplus available for uplift individually and in aggregate as (41). The redistribution prices are then determined by (42), (43), and (44).

$$u_i = z_i^*\delta_i,\ i \in BO^-,\ U = \sum_{i\in BO^-} u_i \tag{40}$$

$$s_i = q_{it}\beta_{it}, i \in LO,\ s_i = z_i^*\delta_i, i \in BO^+$$
$$S = \sum_i s_i, \tag{41}$$

$$z_i^*(\delta_i + \delta_i') = 0, i \in BO^- \tag{42}$$

$$z_i^*\delta_i' = s_iU/S, i \in BO^+ \tag{43}$$

$$q_{it}\beta_{it}' = s_iU/S, i \in LO. \tag{44}$$

In this simple rule, all market participants with positive surplus contribute something to the payments made to block orders requiring additional payments. However, as stated previously, there are many degrees of pricing freedom to derive market-clearing, multi-part prices to fit local market culture and custom. For example, one could easily change the pricing rule above so that only demand bids make lump-sum payments to supply block bids while only supply bids make lump-sum payments to demand block bids. There generally is not a unique set of market-clearing prices.

B. Example from Section II-C Showing Pricing Degrees of Freedom

Consider the optimal solution from the example in Section II-C, with multi-part prices as shown in Table IV. Negative bids are bids to supply, and negative surpluses are losses to that bidder. Recall that in the optimal solution Bid 4 had losses of 1000, and with a lump sum payment of $\delta_4' = 1000$ it can be made whole. Under our proposed *pro rata* rule, Bid 1 and Bid 6 have positive surplus to cover the loss of Bid 4. The allocations under the *pro rata* rule would be Bid 1 pays 444.45, or $\beta_{11}' = 4.4445$ and Bid 6 pays 555.55, or $\beta_{62}' = 3.7$

TABLE IV
MULTI-PART PRICES UNDER PROPOSED *Pro Rata* RULE THAT CLEAR THE MARKET FROM SECTION II-C

Bid	Period 1				
	b_{it}	Q_{it}	β or α	β'	δ' or $\beta' q_{it}$
1	70	100	60	4.4445	$\beta'_{11} q_{11} = 444.45$
2	-5	0	0	0	$\delta'_2 > 5000$
3	-10	0	0	0	0
4	-30	100*	-20	0	$\delta'_3 = -1000$
5	-40	0	0	0	0
6					
λ_t	10				
Surplus	4000				
Bid	Period 2				
	b_{it}	Q_{it}	β or α	β'	δ' or $\beta' q_{it}$
1					
2	-5	0	0	0	0
3					
4	-30	100*	10	0	0
5	-40	50	0	0	0
6	90	150	50	3.7	$\beta'_{62} q_{62} = 555.55$
λ_t	40				
Surplus	8500				

However, it may be the case that the local market culture calls for surplus to be transferred, where possible, only in the hours in which the loss is sustained for Bid 4. In this case, Bid 1 would pay for the loss out of its surplus. The lump-sum payment to Bid 4 is collected from Bid 1 in this case would be $\beta'_{11} * Q_{11} = 10 * 100$.

Having compensated Bid 4 with a payment of 1000 from Bid 1 and Bid 4 does not complete the pricing problem. At prices $\lambda_1 = 10$ and $\lambda_2 = 40$, Bid 2 will want to be dispatched as it has only offered a bid of 5. The reason Bid 2 is not dispatched is that it is quantity infeasible in Period 1. In order to ensure Bid 2 does not self dispatch, it must be sent a discriminatory price of $\delta'_2 \geq 5000$ in order to signal to Bid 2 it should not dispatch due to its quantity infeasible block bid. In equilibrium, δ'_2 does not play any role in redistributing surplus in the market, but the price signal is crucial to ensure the dispatch can be decentralized through the prices.

In short, the degrees of freedom in pricing afforded by the proposed multi-part prices leave a continuum of potential prices that may clear the market. The mechanism by which multi-part prices are determined will undoubtedly have to be agreed upon by the stakeholders participating in the market, just as any mechanism used to determine linear prices as a second-best solution to maximizing surplus would have to be.

VII. SUMMARY, CONCLUSIONS, AND FUTURE WORK

We have shown how, given the bids in a power exchange framework such as APX, one can find the optimal dispatch and calculate revenue adequate, non-confiscatory prices that support the optimal dispatch as a decentralized dispatch. These prices may be non-linear and non-anonymous. However, we have provided a method to maintain uniform, linear prices when feasible, and a method to minimize the deviations from uniform, linear prices when they are not market-clearing while maintaining efficiency. Moreover, we have also shown a practical method by which the proposed discriminatory, multi-part prices can be implemented and have shown through a simple example the pricing degrees of freedom that exist. The framework presented here can be easily extended to include bids for start-up costs and with minimum load restrictions. Also, markets can include all of the constraints imposed by the transmission network. Investments or resource adequacy markets can be designed similarly as is done in [21].

However, the pricing degrees of freedom allowed here may create incentives for unexpected or undesirable bidding behavior that are beyond the scope of this paper. Hence, future work should address market power concerns and bidding incentives resulting from the introduction of non-convex bids that allow market participants more opportunities (degrees of freedom), through increased bidding parameters, to truthfully represent or misrepresent their actual costs. Also in the context of market power and bidding incentives, future work should address the incentive properties for bidding under pricing mechanisms similar to the mechanism proposed here and addressed generally by [22].

APPENDIX
PROOF OF THEOREM 3

Multi-part Price Equilibrium Theorem
There exists a set of revenue adequate, non-confiscatory, multi-part prices for limit orders (λ_t, β'_{it}) and block orders $(\lambda_t, \alpha'_{it})$ or (λ_t, δ'_i), for all i and t that can be constructed from the optimal dual problem defined in (13)-(19) such that the gains from trade are maximized, and the market settlement is revenue adequate and an equilibrium in multi-part prices as defined in Definition 5.

Proof: First, by construction the linearized primal problem as defined in (7)-(12) and its dual problem defined in (13)-(19) of Section III maximizes the gains from trade. Moreover, (8) in the linearized primal problem is identical to Condition 5 in Definition 5 ensuring the market clears.

Step two is to show there is enough market surplus to derive revenue adequate, non-confiscatory prices. We know the market surplus is greater than or equal to zero from the objective function in the dual problem in (13):

$$MS = \sum_{i \in LO} \sum_{t \in T} q_{it}\beta_{it} + \sum_{i \in BO} \sum_{t \in t(i)} z_i^* Q_i \alpha_{it} = \sum_{i \in LO} \sum_{t \in T} q_{it}\beta_{it} + \sum_{i \in BO} z_i^* \delta_i \geq 0. \quad (45)$$

Partitioning the accepted block orders into positive and negative surplus block orders, dropping z_i^* since the surplus is only non-zero for $z_i^* = 1$ and simplifying notation as done in Subsection IV-B, (45) can be restated as

$$MS = \sum_{i \in LO} \sum_{t \in T} q_{it}\beta_{it} + \sum_{i \in BO^+} \delta_i + \sum_{i \in BO^-} \delta_i \geq 0. \quad (46)$$

Rearranging (46) by getting negative surplus values on the right-hand side shows there is enough positive surplus available to compensate accepted block bids with negative surplus:

$$\sum_{i \in LO} \sum_{t \in T} q_{it}\beta_{it} + \sum_{i \in BO^+} \delta_i . \geq - \sum_{i \in BO^-} \delta_i \quad (47)$$

This shows that we can derive revenue adequate prices in the sense of Definition 3. Inserting the prices β'_{it} and δ'_i that satisfy non-confiscation as defined in Definition 2 into (47) yields

$$\sum_{i \in LO} \sum_{t \in T} q_{it}(\beta_{it} - \beta'_{it}) + \sum_{i \in BO+} (\delta_i - \delta'_i) \geq - \sum_{i \in BO-} (\delta_i - \delta'_i). \quad (48)$$

Since $\beta_{it} \geq \beta'_{it} \geq 0$ from (20), $\delta_i \geq \delta'_i \geq 0, \forall i \in BO^+$ from (21), and $0 \geq \delta_i \geq \delta'_i, \forall i \in BO^-$ from (22) in Definition 2, setting $\delta_i = \delta'_i, \forall i \in BO^-$ satisfies non-confiscation and ensures (48) holds thus showing revenue adequate, non-confiscatory prices exist. Stated another way, the set of prices that satisfy (48) must satisfy (23) in Definition 3.

Step three is to show the prices used in the settlement in Definition 4 satisfy revenue adequacy. Summing all settlement payments in Definition 4 over all buyers and sellers results in

$$\sum_{i \in LO \cap B} [q_{it}(\lambda_t + \beta'_{it})] + \sum_{i \in BO \cap B} [z^*_i \sum_{t \in t(i)} (\lambda_t + \alpha'_{it})Q_i] + \sum_{i \in LO \cap S} [-q_{it}(\lambda_t - \beta'_{it})] + \sum_{i \in BO \cap S} [-z^*_i \sum_{t \in t(i)} (\lambda_t - \alpha'_{it})Q_i]. \quad (49)$$

We note that for all accepted block bids $\forall t \in t(i), q_{it} = Q_i$ and $z^*_i = 1$ for accepted bids. We can rearrange (49) to get

$$\lambda_t[\sum_{i \in B} q_{it} - \sum_{i \in S} q_{it}] + \sum_{i \in LO} q_{it}\beta'_{it} + \sum_{i \in BO} \sum_{t \in t(i)} q_{it}\alpha'_{it}. \quad (50)$$

From Condition 5 in Definition 5, $\sum_{i \in B} q_{it} - \sum_{i \in S} q_{it} = 0$, so all that remains is

$$\sum_{i \in LO} q_{it}\beta'_{it} + \sum_{i \in BO} \sum_{t \in t(i)} q_{it}\alpha'_{it}. \quad (51)$$

For revenue adequacy, we must show (51) is non-negative. Partitioning the accepted block orders into positive and negative surplus block orders, and simplifying notation as in Section IV-B, (51) can be restated as

$$\sum_{i \in LO} q_{it}\beta'_{it} + \sum_{i \in BO+} \sum_{t \in t(i)} q_{it}\alpha'_{it} + \sum_{i \in BO-} \sum_{t \in t(i)} q_{it}\alpha'_{it} = \sum_{i \in LO} q_{it}\beta'_{it} + \sum_{i \in BO+} \delta'_i + \sum_{i \in BO-} \delta'_i \quad (52)$$

(52) restates (23) in Definition 3, and because prices are revenue adequate, (52) is non-negative.

The fourth step is to show that prices (λ_t, β'_{it}), for limit orders, and $(\lambda_t, \alpha'_{it})$ or (λ_t, δ'_i) for block orders $\forall i, t$ are equilibrium prices. We proceed by examining limit orders first, then block orders. The proof relies on the properties of the optimal solution to the problems defined by the linearized primal problem in (7) - (12), its dual problem defined in (13) - (19), and our definitions of $\beta'_{it}, \alpha'_{it}$, and δ'_i.

For both buyers and sellers who submit spot limit orders, if a bid is accepted, $q_{it} > 0$. By complementary slackness, for buyers, $b_{it} - \lambda_t = \beta_{it}$; and for sellers, $\lambda_t - b_{it} = \beta_{it}$. From our non-confiscation requirement on β'_{it} from (20), $\beta_{it} \geq \beta'_{it} \geq 0$. Consequently, for buyers, $\lambda_t + \beta'_{it} \leq b_{it}$, i.e. the multi-part market price is less than or equal to the bid; for sellers $\lambda_t - \beta'_{it} \geq b_{it}$, i.e. the multi-part market price is greater than or equal to the bid. Thus, Condition 1) in Definition 5 is satisfied.

For both buyers and sellers who submit spot limit orders, if a bid is rejected, $q_{it} = 0$, and by complementary slackness $\beta_{it} = 0$. For buyers, the bid (willingness to pay) is less than the market price $b_{it} \leq \lambda_t$; and for sellers the bid (willingness to accept) is greater than the market price, $b_{it} \leq \lambda_t$. Thus, Condition 2) in Definition 5 is satisfied.

For buyers who submit block orders, or $i \in BO \cap B$, we have from (16) in the dual problem

$$\sum_{t \in t(i)} (b_{it} - \lambda_t - \alpha_{it})Q_i = \sum_{t \in t(i)} (b_{it} - \lambda_t)Q_i - \delta_i = 0. \quad (53)$$

For a buyer block order that has been accepted, $Q_i = q_{it}, \forall t \in t(i)$ with non-negative surplus, i.e. $i \in BO^+ \cap B$, $\sum_{t \in t(i)} \alpha_{it}Q_i = \delta_i \geq 0$ and from (21)

$$\sum_{t \in t(i)} Q_i\alpha_{it} - \sum_{t \in t(i)} Q_i\alpha'_{it} \geq 0; \quad (54)$$

$$\delta_i - \delta'_i \geq 0. \quad (55)$$

Subsitituting (53) into (54), cancelling terms, and dropping Q_i yields

$$\sum_{t \in t(i)} b_{it} \geq \sum_{t \in t(i)} \lambda_t + \alpha'_{it}. \quad (56)$$

For a buyer block order that has been accepted, $Q_i = q_{it}, \forall t \in t(i)$ with negative surplus, i.e. $i \in BO^- \cap B$, $\sum_{t \in t(i)} \alpha_{it}Q_i = \delta_i \leq 0$ and from (22),

$$0 \geq \sum_{t \in t(i)} Q_i\alpha_{it} - \sum_{t \in t(i)} Q_i\alpha'_{it}; \quad (57)$$

$$0 \geq \delta_i - \delta'_i. \quad (58)$$

Subsitituting (53) into (57), cancelling terms, and dropping Q_i yields

$$\sum_{t \in t(i)} b_{it} \geq \sum_{t \in t(i)} \lambda_t + \alpha'_{it}. \quad (59)$$

Thus, Condition 3) of Definition 5 is satisfied for buyers.
For buyers with block orders that are rejected, $q_{it} = 0 \forall t \in t(i)$, there are no non-confiscation and revenue adequacy constraints on α'_{it} as no transfers take place since $q_{it} = 0$. α'_{it} must be set such that the bid price is less than or equal to the multi-part price, i.e. $\sum_{t \in t(i)} b_{it} \leq \sum_{t \in t(i)} \lambda_t + \alpha'_{it}$. Thus, Condition 4) of Definition 5 is satisfied for buyers.

For sellers who submit block orders, or $i \in BO \cap S$, and have them accepted $Q_i = q_{it}, \forall t \in t(i)$. From (17) in the dual problem,

$$\sum_{t \in t(i)} (-b_{it} + \lambda_t - \alpha_{it})Q_i = \sum_{t \in t(i)} (-b_{it} + \lambda_t)Q_i - \delta_i = 0. \quad (60)$$

For a seller block order with non-negative surplus, i.e. $i \in BO^+ \cap S$, $\sum_{t \in t(i)} \alpha_{it} Q_i = \delta_i \geq 0$ and from (21),

$$\sum_{t \in t(i)} Q_i \alpha_{it} - \sum_{t \in t(i)} Q_i \alpha'_{it} \geq 0; \tag{61}$$

$$\delta_i - \delta'_i \geq 0. \tag{62}$$

Subsitituting (60) into (61), cancelling terms, and dropping Q_i yields

$$\sum_{t \in t(i)} b_{it} \leq \sum_{t \in t(i)} \lambda_t - \alpha'_{it}. \tag{63}$$

For a seller block order with negative surplus, i.e. $i \in BO^- \cap S$, $\sum_{t \in t(i)} \alpha_{it} Q_i = \delta_i \leq 0$ and from (22),

$$0 \geq \sum_{t \in t(i)} Q_i \alpha_{it} - \sum_{t \in t(i)} Q_i \alpha'_{it}; \tag{64}$$

$$0 \geq \delta_i - \delta'_i. \tag{65}$$

Subsitituting (53) into (57), cancelling terms, and dropping Q_i yields

$$\sum_{t \in t(i)} b_{it} \leq \sum_{t \in t(i)} \lambda_t - \alpha'_{it}. \tag{66}$$

Thus, Condition 3) of Definition 5 is satisfied for sellers.

For sellers with block orders that are rejected, $q_{it} = 0$ $\forall t \in t(i)$, there are no non-confiscation and revenue adequacy constraints on α'_{it} as no transfers take place since $q_{it} = 0$. α'_{it} must be set such that the bid price is greater than or equal to the multi-part price, i.e. $\sum_{t \in t(i)} b_{it} \geq \sum_{t \in t(i)} \lambda_t + \alpha'_{it}$. Thus, Condition 4) of Definition 5 is satisfied for sellers.

QED

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