Let $v_h d_h$ and $v_l d_l$ be the reservation prices for heavy and light users, respectively. We show in this appendix that our conclusions are not altered given that $3t < v_h d_h \leq v_l d_l \leq v_h d_h \left(2 - \frac{d_l}{d_h}\right)$. We define $\gamma_h = \frac{v_h d_h}{t}$ and $\gamma_l = \frac{v_l d_l}{t}$.

In this appendix, we also focus our analysis on the case where $K_2 \geq K_1$, and assume that heavy users are sufficiently different from heavy users in terms of their usage rate, or $d_h \geq 4d_l$.

The Monopolist Case

When the monopolist is not capacity-constrained ($K > \bar{d}$), it charges only a flat fee and all consumers who purchase the access service pay the same amount, irrespective of their usage rate. Under the two-part tariff $f$ (fee) and $p \geq 0$ (usage price), light users pay $p_l = f + pd_l$ and heavy users pay $p_h = f + pd_h$ and therefore $p_h \geq p_l$. The monopolist’s optimization problem is formulated as $\max_{(f,p)} (1 - \alpha)x_l(f + pd_l) + \alpha x_h(f + pd_h)$ subject to the supply constraints $0 \leq x_l \leq 1$, $0 \leq x_h \leq 1$ and the positive usage price constraint $p \geq 0$. The solution to this problem yields a flat fee $f = v_l d_l - t$ and no usage price ($p = 0$). In the absence of any capacity constraint, the monopolist is only concerned with penetrating the market profitably, while ensuring that its light user price remains lower than its heavy user price. This means in this case covering entirely the most valuable light user segment (since $3 < \gamma_h \leq \gamma_l$). The optimal price for the monopolist is the maximum price it can charge to the light users while still attracting all of them to buy, or $p_l = v_l d_l - t$, and setting $p_h$ as low as possible, i.e. $p_h = p_l = v_h d_h - t$. Again, it is not surprising that heavy users are charged a lower price than light users as they have a lower reservation price for their ideal service ($v_h d_h \leq v_l d_l$). However, this intuition ignores the mediation of capacity in a firm’s pricing decision. As we show now, the monopolist will use segmented pricing and charge heavy users a higher price than light users even when the latter are willing to pay more than the latter, provided that it is capacity-constrained.
Monopolist With Capacity Constraint

For any given capacity $0 \leq K \leq \bar{d}$, the monopolist must decide what pricing structure to use to engage its limited capacity optimally. For any given $(f, p)$, all light users located to the left of $x_l$ who gain positive surplus will make a purchase, where $x_l = \frac{v_l d_l - f - pd_l}{t}$ is the location of the marginal light users who are just indifferent between buying and not buying. Similarly, all heavy users to the left of $x_h = \frac{v_h d_h - f - pd_h}{t}$ will also make a purchase. Given that the total capacity engaged to service these purchases cannot exceed the total capacity available, the monopolist’s optimization problem is the same as before with the additional capacity constraint $(1 - \alpha) x_l d_l + \alpha x_h d_h \leq K$.

The solution to this optimization problem is illustrated in Figure 1. There we see that when the monopolist’s capacity is sufficiently small, or $K \leq \bar{K}^a = \frac{(\gamma_d d_h - \gamma_h d_l)(1-\alpha)d_l}{2d_h}$, it draws only the light users located nearby, as they are the least resource-demanding and the most profitable customers. The parameter $\bar{K}^a$ is determined by making sure that the constraint $x_h \geq 0$ is binding. To sell all available capacity to light users, the monopolist simply sets $(f, p)$ so that there are just enough light users to exhaust the capacity, or $(1 - \alpha) x_l d_l = K$. This yields $f + pd_l = v_l d_l - \frac{K}{(1-\alpha)d_l}t$. To screen out heavy users, the monopolist makes sure that $f + pd_h$ is greater than $v_h d_h$, or $x_h = 0$, and
that \( f + pd_h \geq f + pd_l \). This is insured by setting \( f + pd_h = \text{Max} \left( v_h d_h, v_l d_l - \frac{K}{(1 - \alpha)d_l} \right) \). When the monopolist’s capacity is smaller than \((\gamma_l - \gamma_h)(1 - \alpha)d_l\), this pricing structure is implemented through a fixed fee \( f = v_l d_l - \frac{K}{(1 - \alpha)d_l} \). As the monopolist’s capacity increases beyond \((\gamma_l - \gamma_h)(1 - \alpha)d_l\) the pricing structure is implemented through a two part-tariff \( f = \frac{(\gamma_l d_h - \gamma_h d_l)(1 - \alpha)d_l - Kd_h}{(d_h - d_l)(1 - \alpha)d_l} t \) and \( p = \frac{K - (\gamma_l - \gamma_h)(1 - \alpha)d_l}{(d_h - d_l)(1 - \alpha)d_l} t \) where \( f > 0 \) and \( p > 0 \). As illustrated in Figure 2, this pricing structure calls on the monopolist to lower its fixed fee to attract more light users as long as the fixed fee is high enough to sift out heavy users \((K \leq (\gamma_l - \gamma_h)(1 - \alpha)d_l)\). As its capacity expands \(((\gamma_l - \gamma_h)(1 - \alpha)d_l \leq K \leq \hat{K}^a)\), the monopolist keeps lowering the fixed fee while simultaneously introducing an increasing usage price component in its pricing structure in order to attract more light users, but still sift out heavy users. This adjustment has the intended effect because a higher usage price hits heavy users harder than it does light users. The combination of these two changes enables the monopolist to deliver more incentives to light users without offering any to heavy users.

As the monopolist’s capacity continues to increase, it pulls in light users that are further and further distant. Eventually, as the capacity increases beyond \(\hat{K}^a\), attracting the heavy users located close to the firm becomes more profitable than attracting additional light users located far away despite the fact that the former use up more capacity. This is when the monopolist adjusts its pricing structure to serve both light and heavy users \(^2\), as illustrated in Figure 1 for \(\hat{K}^a \leq K \leq \hat{K}^b\), where \(\hat{K}^b = \frac{2d - (\gamma_l d_h - \gamma_h d_l)e d_h}{2d} \) and \(d = (1 - \alpha)d_l^2 + \alpha d_h^2\). The optimal mix of light and heavy user are given by optimal sales \(q_l = \frac{(1 - \alpha)}{2} \left( \frac{2Kd_l + (\gamma_l d_h - \gamma_h d_l)e d_h}{d} \right) \) in the light users segment and \(q_h = \frac{\alpha}{2} \left( \frac{2Kd_h - (\gamma_l d_h - \gamma_h d_l)(1 - \alpha)d_l}{d} \right) \) in the heavy user segment. In this case, the monopolist prices its service such that any additional capacity yields the same return whether this incremental capacity is engaged by light users or by heavy users. The two-part tariff that achieves the optimal customer

\(^1\)The quantity \(v_l d_l - \frac{K}{(1 - \alpha)d_l} t\) is smaller than \(v_h d_h\) for \(K \leq (\gamma_l - \gamma_h)(1 - \alpha)d_l\). Also, observe that \((\gamma_l - \gamma_h)(1 - \alpha)d_l \leq \hat{K}^a\) as we have \(v_l d_l \leq v_h d_h(2 - \frac{d_h}{d_l})\) (by assumption). We therefore have that \(\text{Max} \left( v_h d_h, v_l d_l - \frac{K}{(1 - \alpha)d_l} t \right) \) is equal to \(v_l d_l - \frac{K}{(1 - \alpha)d_l} t\) for \(K \leq (\gamma_l - \gamma_h)(1 - \alpha)d_l\) and is equal to \(v_h d_h\) for \((\gamma_l - \gamma_h)(1 - \alpha)d_l \leq K \leq \hat{K}^a\).

\(^2\)Note that when \(K = \hat{K}^a\) the light user price \(v_l d_l - \frac{K}{(1 - \alpha)d_l} t\) equals \(\frac{(\gamma_l + \gamma_h)d_l}{d_h}\) which is smaller than \(v_h d_h\) as we assume that \(v_l d_l \leq v_h d_h(2 - \frac{d_h}{d_l})\). Therefore, the monopolist can start attracting heavy users at \(K = \hat{K}^a\) by setting its heavy user price to \(v_h d_h\).
mix is given by \( f = \frac{\gamma l d - \gamma h d}{2(d_h - d_l)} t \) and \( p = \bar{K}^c - K^t \). By maintaining the level of its fixed fee but decreasing its usage price when it has a larger capacity, the monopolist offers a larger incremental incentive to heavy users to secure more of them.

At a greater capacity (\( \bar{K}^b \leq K < \bar{K}^c \)), where \( \bar{K}^c = \bar{K}^a + \frac{d}{2(d_h - d_l)} \left( \gamma_h \left( 2 - \frac{d}{d_h} \right) - \gamma_l \right) \), all light users have already become the monopolist’s customers. The incremental units of capacity above and beyond \( \bar{K}^b \) are all used to attract additional heavy users among those that remain unserved. The optimal tariff schedule is given by \( f = \frac{K d_i - d + (\gamma_h d_h - \gamma_l d_l) d_i}{d_h - d_l} \) and \( p = \frac{\bar{K}^b - K}{d_h - d_l} t \). In this case, the monopolist continues to lower its usage price all the way to zero (its marginal cost) as its capacity expands in order to attract more heavy users. In the meantime, it raises the fixed fee so that light users are not getting a free ride.

At an even greater capacity (\( K \geq \bar{K}^c \)), the monopolist can no longer lower its usage price and attract more heavy users as the usage price has been lowered all the way to zero when the capacity reached the level \( \bar{K}^c \). In this case, the monopolist charges a flat fee \( f^* \), and the level of the optimal flat fee depends on the relative size of the heavy and light user segments as measured by \( \alpha \). There are three possible cases and we consider them in turn. First, if the light user segment size relative
to the heavy user segment is large (\( \alpha \leq \frac{1}{2\gamma_l-\gamma_h-1} \)), then the monopolist has no incentive to lower its flat fee and engage additional capacity in the heavy user segment. It charges \( f^* = v_l d_l - t \) and is left with unused capacity although the market is not covered. Second, if the size of the light user segment relative to the heavy user segment is intermediate (\( \frac{1}{2\gamma_l-\gamma_h-1} \leq \alpha \leq \frac{1}{\gamma_h-1} \)), then the monopolist will have the incentive to lower its flat fee and engage additional capacity in the heavy user segment, up to a maximum level \( K^* = \frac{\gamma_h a d_h - (1-\alpha)(d_h-2t)}{2} \) lower than \( \bar{d} \). The flat fee charged by the monopolist will depend on its capacity. If the monopolist’s capacity is lower than \( K^* \) (the monopolist is capacity constrained), then the optimal fixed fee is \( f^* = v_l d_h - \frac{K-(1-\alpha)d_l}{a d_h} t \). If its capacity is greater than \( K^* \), then the optimal fixed fee is \( f^* = \frac{\gamma_h a-(1-\alpha)}{2a} t \) (the monopolist has unused capacity and the market is not covered). Third, and finally, if the light user segment size relative to the heavy user segment is small (\( \alpha \geq \frac{1}{2\gamma_l-\gamma_h-1} \)), then the monopolist will have the incentive to lower its flat fee all the way to \( v_h d_h - t \) to attract all the remaining heavy users. The flat fee charged by the monopolist will also depend on its capacity. It is \( f^* = v_h d_h - \frac{K-(1-\alpha)d_l}{a d_h} t \) if the monopolist’s capacity is lower than \( \bar{d} \) (the monopolist is capacity constrained), and it is \( f^* = \frac{\gamma_h a-(1-\alpha)}{2a} t \) if its capacity is greater than \( \bar{d} \) (the monopolist is not capacity constrained and the whole market is covered).

The Competitive Case

To analyze the competitive case, note that at any given prices \((p^2_l, p^1_l)\) charged respectively by the two competing firms, the light users who are indifferent between purchasing from either firm must be located at \( \bar{x}_h = \frac{p^2_l - p^1_l + t}{2t} \). These indifferent light users may derive negative surplus at those prices. In that case, they do not purchase from any firm and the light user segment is uncovered, i.e. \( p^1_l + p^2_l > 2v_l d_l - t \). Or they derive zero surplus from their purchase such that the light user segment is just covered (just covered), i.e. \( p^1_l + p^2_l = 2v_l d_l - t \). Or they enjoy positive surplus such that the light user segment is not only covered, but also competitive, i.e. \( p^1_l + p^2_l < 2v_l d_l - t \). We now take up each possibility in turn to look for an equilibrium.
Light User Segment Uncovered

When the light user segment is uncovered, both firms are local monopolies in the heavy user segment. From our analysis of the monopolist case, we know that in any such equilibrium $x_i^l$ is strictly greater that $x_i^h$. Therefore, if the light user segment is uncovered, so must be the heavy user segment. Thus, both firms will price their access services as if they were monopolists. We first show that in any such equilibria, both firms can’t be present in both markets, and then derive the two possible equilibria where firms are local monopolists in their markets.

In any equilibrium where the light user segment is uncovered, we have $x_1^l + x_2^l < 1$ and therefore $x_i^l$ must be smaller than $\frac{1}{2}$ for at least one of the two firms $i = 1, 2$ for this inequality to be possible. At least one firm must be serving less than half of the light user segment, say firm 1, and is engaging less than $\frac{(1-\alpha)d_i}{2}$ units of capacity in the light user markets. From our analysis of the monopolist case, we know that the capacity threshold $\bar{K}^a$ below which firm 1 will exclusively serve the light user segment is greater than $\frac{(1-\alpha)d_i}{2}$. Thus firm 1 is engaging less than $\bar{K}^a$ units of capacity in the light user segment and must therefore be serving this segment exclusively.

They are therefore only two possible types of equilibrium. In the first type of equilibria, each of the firms has a smaller capacity than the capacity threshold $\bar{K}^a$. Each firm serves exclusively the light user segment and enjoys a local monopoly in the light user segment. This equilibrium is shown in region 1 of Figure 1. The analysis and results of the monopolist case apply here. The necessary and sufficient conditions for this first type of equilibrium are $K_i \leq \bar{K}^a$ for $i = 1, 2$ and $K_1 + K_2 < (1-\alpha)d_i$.

In the second type of equilibria, only the small capacity firm 1 has a smaller capacity than the capacity threshold $\bar{K}^a$. Firm 1 serves exclusively the light user segment and enjoys a local monopoly in the light user segment. Firm 2 has a larger capacity than the capacity threshold $\bar{K}^a$ and is present in both markets. Firm 2 charges a two-part tariff with a flat fee $f = \frac{(v_l-v_h)d_i d_o}{2(d_h-d_l)}$ and a per-unit price $p = (\bar{K}^c - K)\frac{d}{d}$. This equilibrium is shown in region 2 of Figure 1. In this second type of equilibrium, we must have $x_1^l + x_2^l < 1$. From the monopolist analysis, we know that
Figure 1: Equilibrium with Two Competing Firms

\[ x_l^1 = \frac{K_1}{(1-\alpha)d_l} \quad \text{and} \quad x_l^2 = \frac{2K_2d_l + (\gamma_l d_h - \gamma_h d_l)\alpha d_h}{2d} \]. The necessary and sufficient conditions for this second type of equilibrium are

\[
\overline{K}^a \leq K_2 \leq \left(1 - \frac{K_1}{(1-\alpha)d_l}\right) \frac{\hat{d}}{d_l} - \left(\gamma_l d_h - \gamma_h d_l\right)\frac{\alpha d_h}{2d_l}
\]

which can be rewritten as

\[
\overline{K}^a \leq K_2 \quad \text{and} \quad K_1 \leq (1-\alpha)d_l \left(1 - \frac{2K_2d_l + (\gamma_l d_h - \gamma_h d_l)\alpha d_h}{2d}\right)
\]

**Light User Segment Just Covered**

In any equilibrium where the light segment is just covered, the heavy user segment can be either uncovered, or just covered, or competitive. We will first show that there exists no equilibria where the heavy user segment is either just covered or competitive or fully served by one of the firms. We will then show that there exists no equilibria where the heavy user segment is served by both firms. As a result, in any equilibrium where the light user segment is just covered, the smallest firm 1 is present in the light user segment only, while the largest firm 2 is present in both the light
and heavy user segments. We will show that in such an equilibrium, the smallest firm 1 must be capacity constrained, while the largest firm 2 may or may not be capacity constrained, and in the case where it is not capacity constrained, it must be charging a flat fee. We proceed first by proving a useful lemma.

**Lemma 1** In any equilibrium where the light segment is just covered, at least one of the firms must be capacity constrained.

**Proof:** Suppose to the contrary that there exists an equilibrium where the light user segment is just covered and none of the firms is capacity constrained. An unconstrained firm can always lower its light users price $p_i^l$ by $\epsilon$ in order to attract additional light users and engage some of its unused capacity. As the light user segment is just covered, the decrease in price to light users will attract $(1 - \alpha) \frac{t}{2t}$ additional light users, with a net gain of $(1 - \alpha) \frac{t}{2t} (p_i^l - \epsilon) - (1 - \alpha) x_i^l \epsilon$. Since the light user segment is just covered, the indifferent light user derives zero surplus $p_i^l = v_l d_l - x_i^l t$, and the first order net gain can be simplified to $(1 - \alpha) \frac{t}{2t} (v_l d_l - 3 x_i^l t)$. To sustain the equilibrium, the first order gain is necessarily negative, i.e. $v_l d_l \leq 3 x_i^l t$. Summing these inequalities for firm 1 and firm 2, we obtain that $2v_l d_l \leq 3(x_1^l + x_2^l) t$. As the light user segment is just covered, $x_1^l + x_2^l = 1$ and therefore $\gamma_l = \frac{w_i d_i}{t} \leq \frac{3}{3}$; but by assumption we have $\gamma_l \geq 3$. A contradiction. Q.E.D.

We proceed now to show that in any equilibrium where the light segment is just covered, the heavy user segment is necessarily uncovered (i.e. neither just covered, nor competitive, nor fully served by one of the firms). In any such equilibrium, as the light user segment is just covered, the indifferent light users derive zero surplus from their purchase, i.e. $p_1^l + p_2^l = 2v_l d_l - t$. As prices in the light user segment are lower than prices in the heavy user segment, and since $v_l d_l \geq v_h d_h$ (by assumption), we must have in equilibrium

$$p_h^1 + p_h^2 \geq p_1^l + p_2^l = 2v_l d_l - t \geq 2v_h d_h - t.$$

(0.2)

The heavy users segment can’t be competitive since inequality (0.2) shows that the indifferent heavy users is deriving a non positive surplus ($p_h^1 + p_h^2 \geq 2v_h d_h - t$). If the heavy user segment were just covered, the indifferent heavy users would be deriving zero surplus. We would have
\( p_h^1 + p_h^2 = 2v_h d_h - t \), and with inequality (0.2) this would imply that \( v_l d_l = v_h d_h \). This is the case treated in the main paper, and we showed there that there exist no equilibrium where both the light and the heavy user segments are just covered. As a result, the indifferent heavy users must be deriving negative surplus since \( p_h^1 + p_h^2 > 2v_h d_h - t \). This proves that there exists no equilibria where the heavy user segment is either just covered or competitive. Finally, it is easy to check that there exists no equilibria where the heavy user segment is fully served by one of the firms. Suppose to the contrary that there exists an equilibrium where the heavy user segment is fully served by one of the firms. In any such equilibrium, the firm that is fully serving the heavy user segment would charge \( p_h^i = v_h d_h - t \) to the heavy users and \( p_l^i = v_l d_l - x_l^i t \) to the light users. As \( p_l^i \leq p_h^i \), this would imply that \( \gamma_h - \gamma_l \geq 1 - x_l^i > 0 \), but we have \( \gamma_h \leq \gamma_l \) (by assumption). Contradiction. \( Q.E.D. \)

We now show, in the following Lemma 2, that there exists no equilibrium where the heavy user segment is served by both firms:

**Lemma 2** In any equilibrium where the light user segment is just covered, only one of the firms is present in the heavy user segment

**Proof:** Suppose to the contrary that both firms are present in the heavy user segment. A firm can always raise its price to the heavy users by \( \epsilon \) and lower its price to the light users by \( \epsilon t \). As the heavy user segment is not competitive, such price changes will drop \( \alpha \epsilon d_h \) heavy users and free \( \alpha \epsilon d_h \) units of capacity. As the light user segment is just covered, the decrease in price to the light users will attract \( (1 - \alpha) \frac{\epsilon'}{2} \) light users and engage \( (1 - \alpha) \frac{\epsilon'}{2} d_l \) units of additional capacity in that segment. Such simultaneous price changes are always feasible, irrespective of whether or not the firm is capacity-constrained, as it can always set \( \alpha \epsilon d_h = (1 - \alpha) \frac{\epsilon'}{2} d_l \). The net gain from these price changes is \( \alpha(x_h^i - \frac{\epsilon'}{2}) \epsilon - \alpha \epsilon p_h^i + (1 - \alpha) \frac{\epsilon'}{2} (p_l^i - \epsilon t) - (1 - \alpha) x_l^i \epsilon t \). As none of the user segments are competitive, we must have \( p_h^i = v_h d_h - x_h^i t \) and \( p_l^i = v_l d_l - x_l^i t \). Then, the first order net gain can be simplified to \( \alpha \frac{\epsilon}{2d_l} (2x_h^i d_l - 3x_l^i d_l - v_h d_h d_l + v_l d_l d_h) \). To sustain the equilibrium, the first order gain is necessarily negative, which implies \( 2x_h^i d_l - 3x_l^i d_h \leq \gamma_h d_l - \gamma_l d_h \). As both firms are present
in the heavy user segment \((x_h^i > 0)\), this implies

\[
3x_h^i d_h > \gamma_l d_h - \gamma_h d_l \quad i = 1, 2.
\] (0.3)

As the light user segment is just covered, we have \(x_l^i + x_h^i = 1\). Summing up the inequalities (0.3) for \(i = 1, 2\) we obtain \(2\gamma_l < 3 + 2\gamma_h \frac{d_l}{d_h}\). Given that \(\gamma_h \leq \gamma_l\) (by assumption) and \(d_h \geq 4d_l\) (by assumption), we get that \(2\gamma_h < 3 + \frac{3\gamma_l}{2}\) and thus \(\gamma_h < 2\). But \(\gamma_h\) is greater than 3 by assumption. A contradiction. Q.E.D.

We know from Lemma 1 that in any equilibrium where the light segment is just covered, one of the firms must be capacity constrained. We now show that if only one of the firms is capacity constrained, then the unconstrained firm is the one that is present in the heavy user segment. Suppose to the contrary that there exists an equilibrium where the constrained firm \(i\) is present in the heavy user segment. As firm \(i\) is present in the heavy user segment, firm \(-i\) is not (from Lemma 2). We first focus on the unconstrained firm \(-i\). It can always lower its light users price \(p_l^{-i}\) by \(\epsilon\) in order to attract additional light users and engage some of its unused capacity. As the light user segment is just covered, the decrease in price to light users will attract \((1 - \alpha)\frac{\epsilon}{d_l}\) additional light users, with a net gain of \((1 - \alpha)\frac{\epsilon}{d_l}(p_l^{-i} - \epsilon) - (1 - \alpha)x_l^{-i}\epsilon\). Since the light user segment is just covered, the indifferent light user derives zero surplus \(p_l^{-i} = v_l d_l - x_l^{-i}\epsilon\), and the first order net gain can be simplified to \((1 - \alpha)\frac{\epsilon}{d_l}(v_l d_l - 3x_l^{-i}\epsilon\)). To sustain the equilibrium, the first order gain is necessarily negative, and this implies

\[
\gamma_l \leq 3x_l^{-i}\]

(0.4)

We now focus on the constrained firm \(i\). As we argued in the proof of Lemma 1, the firm \(i\) that is present in both segment can always raise its price to the heavy users by \(\epsilon\) and lower its price to the light users by \(\epsilon t\); in equilibrium, it will not have the incentive to do so, and this implies that the inequality (0.3) is satisfied: \(3x_l^i d_h > \gamma_l d_h - \gamma_h d_l\). Dividing this inequality by \(d_h\), adding it to the inequality (0.4), and observing that \(x_l^i + x_l^{-i} = 1\), we obtain \(2\gamma_l - \gamma_h \frac{d_l}{d_h} < 3\). Now, given that \(\gamma_h \leq \gamma_l\) (by assumption) and \(d_h \geq 4d_l\) (by assumption), we get that \(2\gamma_h < 3 + \frac{3\gamma_l}{2}\) and thus \(\gamma_h < 2\). But \(\gamma_h\) is greater than 3 by assumption. A contradiction. Q.E.D.
It is now easy to see that the small capacity firm 1 is the firm that is only present in the light user segment. To prove this claim, we show that if firm $i$ is present in the light user segment only, we necessarily have that $x_{l}^{-i} < \frac{1}{2}$. If firm $i$ is present in the light user segment only, then firm $i$ is present is both segments, and therefore, as argued in the proof of Lemma 1, the inequality (0.3) must be satisfied: $3x_{l}^{i}d_{h} > \gamma_{l}d_{h} - \gamma_{h}d_{l}$. Suppose to the contrary that $x_{l}^{-i} \geq \frac{1}{2}$, then since $x_{l}^{i} + x_{l}^{-i} = 1$ this would imply $x_{l}^{i} < \frac{1}{2}$ and therefore the inequality (0.3) would yield $\frac{3}{2}d_{h} > \gamma_{l}d_{h} - \gamma_{h}d_{l}$, which can be rewritten as $2\gamma_{l} < 3 + 2\gamma_{h} \frac{d_{l}}{d_{h}}$. Given that $\gamma_{h} \leq \gamma_{l}$ (by assumption) and $d_{h} \geq 4d_{l}$ (by assumption), we get that $2\gamma_{l} < 3 + \frac{2\gamma_{h}}{2}$ and thus $\gamma_{h} < 2$. But $\gamma_{h}$ is greater than 3 by assumption. A contradiction. 

Q.E.D.

As a corollary, we obtain that in any equilibrium where the light user segment is just covered and where only one of the firm is capacity constrained, the large capacity firm 2 is the unconstrained firm, and the small capacity firm 1 is the constrained one. We now show that if the largest firm 2 is unconstrained, it necessarily charges a flat fee. Suppose to the contrary that the unconstrained firm 2 charges a two-part tariff, i.e. $p_{l}^{2} < p_{h}^{2}$. Firm 2 would then have a clear incentive to lower its heavy users price and expand in the uncovered heavy users segment (since $\gamma_{h} > 3$ by assumption). Q.E.D.

We have proved so far that in any equilibrium where the light user segment is just covered, the heavy user segment is not covered and is served by the large capacity firm 2 only. In addition, the small capacity firm 1 is necessarily constrained, and if the large capacity firm 2 is not constrained it charges a flat fee. The only possible equilibria where the light user segment is just covered are therefore of either two sorts:

- both firms are capacity constrained, charge a two-part tariff, and only the large capacity firm 2 is present in the heavy user segment: this equilibrium is shown in region 3 of Figure 1;

- the small capacity firm 1 is constrained, charges a two-part tariff and is present in the light user segment only, while the large capacity firm 2 is not constrained, charges a flat fee and is present in both segments of the market: this equilibrium is shown in region 4 of Figure 1.
The boundary locus between region 3 and 4 would yield an equilibrium where the small capacity firm 1 is constrained, charges a two-part tariff and is present in the light user segment only, while the large capacity firm 2 is just constrained, charges a flat fee $f^2$ and is present in both segments of the market. As firm 1 is capacity constrained and is present in the light user segment only, we have $x^1_l = \frac{K_1}{(1-\alpha)d_l}$. As the light users segment is just covered, this implies $x^2_l = 1 - \frac{K_1}{(1-\alpha)d_l}$. As none of the user segments are competitive and firm 2 is charging a flat fee, we must have

$$\alpha \epsilon f^2 \geq x^2_l = 1 - \frac{K_1}{(1-\alpha)d_l}.$$

As none of the user segments are competitive and firm 2 is charging a flat fee, we must have $f^2 = v_h d_h - x^2_h t$ and $f^2 = v_l d_l - x^2_l t$, and hence $x^2_h = x^2_l - (\gamma_l - \gamma_h) = 1 - \frac{K_1}{(1-\alpha)d_l} - (\gamma_l - \gamma_h)$. The capacity engaged by firm 2 is $(1-\alpha)x^2_l d_l + \alpha x^2 h d_h = \bar{d} \left( 1 - \frac{K_1}{(1-\alpha)d_l} \right) - (\gamma_l - \gamma_h) \alpha d_h$. At the boundary between region 3 and 4, the capacity engaged by firm 2 is exactly equal to $K_2$, and we have $K_2 = \bar{d} \left( 1 - \frac{K_1}{(1-\alpha)d_l} \right) - (\gamma_l - \gamma_h) \alpha d_h$. But, as we move to region 3, firm 2 becomes capacity constrained and we necessarily have

$$K_2 < \bar{d} \left( 1 - \frac{K_1}{(1-\alpha)d_l} \right) - (\gamma_l - \gamma_h) \alpha d_h \quad (0.5)$$

We now proceed to check that these are indeed equilibria of the game, and determine the sufficient conditions for these equilibria to exist. We first, consider the region 3 equilibria, where the light user segment is just covered, both firms are capacity constrained, charge a two-part tariff, and only the large capacity firm 2 is present in the heavy user segment. To check the sufficient conditions for this equilibrium, note that firm 2 can always raise its price to the heavy users by $\epsilon$ and lower its price to the light users by $\epsilon t$. As the heavy user segment is not covered, such price changes will drop $\frac{\alpha \epsilon}{t}$ heavy users and free $\alpha \epsilon t d_h$ units of capacity. As the light user segment is just covered, the decrease in price to the light users will attract $\frac{(1-\alpha)\epsilon t}{2\epsilon}$ light users and engage $(1-\alpha)\frac{\epsilon t}{2\epsilon} d_l$ units of additional capacity in that segment. Such simultaneous price changes are always feasible, as firm 2 optimally sets $\alpha \epsilon t d_h = (1-\alpha)\frac{\epsilon t}{2\epsilon} d_l$. The net gain from these price changes is $\alpha(x^2_l - \epsilon)p^2_l + (1-\alpha)\frac{\epsilon t}{2\epsilon}(p^2_l - \epsilon t) - (1-\alpha)x^2_l \epsilon t$. As none of the user segments are competitive, we must have $p^2_h = v_h d_h - x^2_h t$ and $p^2_l = v_l d_l - x^2_l t$. As firm 1 is capacity constrained and is present in the light user segment only, we have $x^1_l = \frac{K_1}{(1-\alpha)d_l}$. As the light users segment is just covered, this implies $x^2_l = 1 - \frac{K_1}{(1-\alpha)d_l}$. As firm 2 is also capacity constrained, we have $x^2_h = \frac{K_1 + K_2 - (1-\alpha)d_l}{\alpha d_h}$. 

\[12\]
We can therefore express the net gain as a function of $K_1$ and $K_2$. To sustain the equilibrium, the first order gain is necessarily negative, which implies:

$$K_1 \leq (1 - \alpha)d_t \left( 1 - \frac{2K_2d_t + (\gamma d_h - \gamma h d_t)\alpha d_h}{2d + \alpha d_h^2} \right) \quad \text{(0.6)}$$

On the other hand, note that firm 1 will never lower its price to attract more light users since it is already capacity-constrained. And as we show now, neither will it have any incentive to raise its price charged to light users to make room for some heavy users. In any equilibrium where the light users segment is just covered we have $K_1 + K_2 \geq (1 - \alpha)d_t$. When this is combined with the region 3 inequality (0.6), we obtain that $K_1 \leq \frac{(1-\alpha)d_t}{3} \left( 3 - \gamma_l + \gamma h \frac{d_t}{d_h} \right)$. Given that $\gamma_h \leq \gamma_l$ (by assumption) and $d_h \geq 4d_t$ (by assumption), we get that $3 - \gamma_l + \gamma h \frac{d_t}{d_h} \leq 3 - \gamma_h + \frac{d_t}{d_h} = 3 - \frac{3\gamma_h}{4}$. Finally, as $\gamma_h > 3$ this implies that $3 - \gamma_l + \gamma h \frac{d_t}{d_h} \leq \frac{3}{4}$, and therefore $K_1 \leq \frac{(1-\alpha)d_t}{4}$. As $\bar{K}^a \geq \frac{(1-\alpha)d_t}{3}$ this implies that $K_1 \leq \bar{K}^a$, and therefore that firm 1 has no incentive to deviate from the proposed equilibrium. Q.E.D.

Alternatively, as it is charging a two-part tariff, firm 2 can also raise its price to the light users by $\epsilon$ and lower its price to the heavy users by $\epsilon'$. As the light user segment is just covered, the increase in price to the light users will drop $\frac{(1-\alpha)d_t}{T}$ light users and free $(1-\alpha)\frac{d_t}{T}$ units of additional capacity in that segment. As the heavy user segment is not fully served, such price changes will attract $\alpha\frac{\epsilon'}{T}$ heavy users and engage $\alpha\frac{\epsilon'}{T}d_h$ units of capacity. Such simultaneous price changes are always feasible, as firm 2 optimally sets $\alpha\frac{\epsilon'}{T}d_h = (1-\alpha)\frac{d_t}{T}$. The net gain from these price changes is $(1-\alpha)(x_l^2 - \frac{\epsilon}{T})\epsilon - (1-\alpha)\frac{\epsilon'}{T}p_l^2 + \alpha\frac{\epsilon'}{T}(p_h^2 - \epsilon) - \alpha x_h^2 \epsilon'$. Again, we can express the net gain as a function of $K_1$ and $K_2$. To sustain the equilibrium, the first order gain is necessarily negative, which implies:

$$K_1 \leq (1 - \alpha)d_t \left( 1 - \frac{2K_2d_t + (\gamma l d_h - \gamma h d_t)\alpha d_h}{2d + \alpha d_h^2} \right) \leq K_1 \quad \text{(0.7)}$$

Inequalities (0.5), (0.6) and (0.7) define the boundaries of region 3.

Second, consider the region 4 equilibria, where the light user segment is just covered, the small capacity firm 1 is capacity constrained, charges a two-part tariff and is present in the light user segment only, while the large capacity firm 2 is not capacity constrained, charges a flat fee $f_2^2$, and
is present in both the light and heavy user segments.

As the light users segment is just covered, we have that \( f^2 = vl dt - x_l^2 t = vl dt - (1 - x_l^1) t \). As firm 1 is capacity constrained, we have that \( x_l^1 = \frac{K_1}{d_l} \), and therefore \( f^2 = vl dt - (1 - \frac{K_1}{(1-\alpha)d_l}) t \). Also, we have that \( f^2 = vh dh - x_h^2 t \), hence \( x_h^2 = 1 - \frac{K_1}{(1-\alpha)d_l} - (\gamma_l - \gamma_h) \). As the largest firm 2 is not capacity constrained, we have that \((1-\alpha)x_l^2 dt + \alpha x_h^2 dh < K_2\), hence the condition

\[
K_1 > (1 - \alpha)d_l \left(1 - \frac{K_2 + (\gamma_l - \gamma_h)dh}{d} \right) \quad (0.8)
\]

To check the sufficient conditions for this equilibrium, note that firm 2 has no incentive to raise its price since it desires to serve the whole market as a monopolist, but it may want to lower its price to either light users or both light and heavy users in order to gain a larger market share. The most profitable way for firm 2 to lower its price is to lower its price to both segments of consumers by the same amount, i.e. reducing its flat fee.\(^3\) In that case, if firm 2 lowers its flat fee by \( \epsilon > 0 \), it gains \( \frac{(1-\alpha)\epsilon}{dt} \) light users, each of whom pays \((f_2 - \epsilon)\). In addition, it gains \( \frac{\alpha \epsilon}{t} \) heavy users, each of whom also pays \((f_2 - \epsilon)\). However, firm 2 incurs the total loss of \((1 - x)\epsilon\), as it charges a lower price to all those light and heavy users who are currently buying from it. The net gain is \((1-\alpha)\frac{d_l}{2}(f_2 - \epsilon) - (1-\alpha)x_l^2 \epsilon + \alpha x_h^2 \epsilon\). As \( f_2 = vl dt - (1 - \frac{K_1}{(1-\alpha)d_l}) t \), we can express the first order gain as a function of \( K_1 \). To sustain the equilibrium, the first order gain is necessarily negative, which implies:

\[
K_1 \leq (1 - \alpha)d_l \left(1 - \frac{1 + 3\alpha \gamma_l - 2\alpha \gamma_h}{3 + \alpha} \right) \quad (0.9)
\]

Firm 1, on the other hand, will never lower its price to attract more light users since it is already capacity-constrained. It is also easy to check that \( K_1 < \tilde{K}^a \) and that firm 1 therefore has no incentive to raise its price charged to light users to make room for some heavy users. To check that, observe that the quantity \( \frac{(1+3\alpha \gamma_l - 2\alpha \gamma_h)}{3+\alpha} \) in the right hand side of inequality (0.9) is greater than \( \frac{1}{2} \) since \( 3 \leq \gamma_h \leq \gamma_l \). This implies that \( K_1 \leq \frac{(1-\alpha)dl}{2} < \tilde{K}^a \). Thus, firm 1 has no incentive to deviate from the proposed equilibrium. \( Q.E.D. \)

\(^3\)The alternative is for the firm to use a two-part tariff to deliver a lower price to light users than to heavy users. However, such a deviation can be shown to be less profitable, given that the demand in the light user market is at a kink.
**Light User Segment Competitive**

We will show in this section that in any equilibrium where the light user segment is competitive, none of the firms is capacity constrained. We will then establish that the only possible equilibrium is the standard Hotelling equilibrium whereby each firm charges the Hotelling price $t$ in each of the light and heavy user segments. We will determine the sufficient conditions for the existence of such an equilibrium. This equilibrium is shown in region 5 of Figure 1.

The two following lemmas (Lemma 3 and Lemma 4) establish that in any equilibrium where the light user segment is competitive, none of the firms can be capacity-constrained.

**Lemma 3** *There exists no equilibrium where the light user segment is competitive and exactly one firm is capacity constrained*

**Lemma 4** *There exists no equilibrium where the light user segment is competitive and both firms are capacity constrained*

*Proof of Lemma 3:* Suppose to the contrary that there exists an equilibrium where the light user segment is competitive and exactly one firm, say firm $-i$ is capacity constrained. Since the light user segment is competitive, the indifferent light users derive positive surplus. The unconstrained firm $i$ must therefore be charging a flat fee $f^i$, otherwise it would be able to profitably deviate by raising its price to the light users, extract additional surplus from its light users without loosing any of them to the capacity constrained firm $-i$. Also, it must be the case that the indifferent heavy users are deriving non positive surplus, otherwise the unconstrained firm $i$ would be able to profitably deviate by raising its price to the heavy users, extract additional surplus from its heavy users without loosing any of them to the capacity constrained firm $-i$. So there are two possibilities: either the indifferent heavy users are deriving negative surplus and the heavy users segment is uncovered; or the indifferent heavy users are deriving zero surplus and the heavy users segment is just covered. We now take up each possibility in turn to reach a contradiction.
First, suppose that the heavy users segment is uncovered. Firm $i$ can lower its flat fee by $\epsilon$ (to both types of users), attract $(1-\alpha)\frac{x^i}{2t}$ light users and $\alpha \frac{x^i}{T}$ heavy users, with a net gain of $(1-\alpha)\frac{x^i}{2t}(f^1-\epsilon) - (1-\alpha)x^i_1 \epsilon + \alpha \frac{x^i}{T}(f^1-\epsilon) - \alpha x^i_h \epsilon$. Alternatively, firm $i$ can raise its flat fee by $\epsilon$, drop $\alpha \frac{x^i}{T}$ heavy users but none of its light users (who can’t be served by the capacity constrained competitor), with a net gain of $(1-\alpha)x^1_l \epsilon + \alpha(x^i_h - \frac{x^i}{T})\epsilon - \alpha \frac{x^i}{T}f^1$. Firm $i$ will not have the incentive for any of those two possible deviations if and only if $f^1 \leq \frac{2t}{1+\alpha}((1-\alpha)x^1_l + \alpha x^i_h)$ and $f^1 \geq \frac{1}{\alpha}(1-\alpha)x^1_l + \alpha x^i_h)$. Combining these two inequalities we obtain that $\alpha > 1$. A contradiction.

Second, suppose that the heavy users segment is just covered. Firm $i$ can lower its flat fee by $\epsilon$ (to both types of users), attract $(1-\alpha)\frac{x^i}{2t}$ light users and $\alpha \frac{x^i}{T}$ heavy users, with a net gain of $(1-\alpha)\frac{x^i}{2t}(f^1-\epsilon) - (1-\alpha)x^i_1 \epsilon + \alpha \frac{x^i}{T}(f^1-\epsilon) - \alpha x^i_h \epsilon$. Firm $i$ will have no incentive for such a deviation if and only if $f^i \leq 2x_it$. As the indifferent heavy users are deriving zero surplus, we have that $f^i = v_h d_h - x_h t$. Combined with the previous inequality this yield $v_h d_h \leq x_h t + 2x_it$ and therefore $v_h d_h \leq 3t$, whereas we have assumed that $3t < v_h d_h$. A contradiction. Q.E.D.

Proof of Lemma 4: Suppose to the contrary that there exists an equilibrium where the light user segment is competitive and both firms are capacity constrained. Then it must be the case that both firms charge a flat fee $f^i$. Otherwise, if firm $i$ were charging a two-part tariff, it would profitably deviate by raising its light users price, extracting additional surplus from its light users, without loosing any of them to the capacity constrained firm $-i$ ($i=1,2$). It also must be the case that the indifferent heavy users have non positive surplus. Otherwise any of the two firms would profitably deviate by raising its heavy users price, extracting additional surplus from its heavy users, without loosing any of them to the capacity constrained competitor. We therefore have that $x^i_l = \frac{f^i-f^i+t}{2t}$ and $x^i_h = \frac{v_h d_h - f^i}{t}$. Any of the two firms, say firm $i$ can always raise its flat fee by $\epsilon$ to both types of users. It will drop $\frac{x^i}{T}$ heavy users but none of its light users (who can’t be served by the capacity constrained firm $-i$), with a net gain of $(1-\alpha)x^i_1 \epsilon + \alpha(x^i_h - \frac{x^i}{T})\epsilon - \alpha \frac{x^i}{T}f^i$. To sustain the equilibrium, the first order gain is necessarily negative, i.e. $\alpha f^i \geq (\alpha x^i_h + (1-\alpha)x^i_l)t$. Summing up these inequalities for firm 1 and 2, and using the fact that $x^1_l + x^2_l = 1$ and $x^1_h t = v_h d_h - f^i$, we
obtain that

\[ 2\alpha(f^1 + f^2) \geq 2\alpha v_h d_h + (1 - \alpha)t \]  

(0.10)

Alternatively, any of the two firms, say firm \( i \) can always lower its price to the light users by \( \epsilon \) and increase its price to the heavy users by \( \epsilon' \). As the heavy user segment is not competitive, such price changes will drop \( \frac{\alpha \epsilon'}{t} \) heavy users and free \( \alpha \frac{\epsilon'}{t} d_h \) units of capacity. As the light user segment is competitive, the decrease in price to the light users will attract \( (1 - \alpha) \frac{\epsilon}{2t} \) light users and engage \( (1 - \alpha) \frac{\epsilon}{2t} d_l \) units of additional capacity in that segment. Such simultaneous price changes are always feasible, as firm \( i \) optimally sets \( \alpha \frac{\epsilon'}{t} d_h = (1 - \alpha) \frac{\epsilon}{2t} d_l \). The net gain from these price changes is \((1 - \alpha) \frac{\epsilon'}{2t}(f^i - \epsilon) - (1 - \alpha)x^l_i \epsilon + \alpha(x^h_i - \frac{\epsilon}{2}) \epsilon' - \alpha \frac{\epsilon}{2} f^i \). To sustain the equilibrium, the first order gain is necessarily negative, i.e. \( f^i(d_h - d_l) \leq 2x^l_i t d_h - x^h_i t d_l \). Summing up these inequalities for firm 1 and firm 2, and using the fact that \( x^l_1 + x^l_2 = 1 \) and \( x^h_i t = v_h d_h - f^i \), we obtain that

\[ (f^1 + f^2)(d_h - 2d_l) \leq 2td_h - 2v_h d_h d_l \]  

(0.11)

The inequalities (0.10) and (0.11) imply that \( v_h d_h \leq -\frac{1 - \alpha}{2\alpha} (1 - \frac{2d_h}{d}) t \). As \( d_h \geq 4d_l \) (by assumption), this implies that \( v_h d_h \leq 3t \). Contradiction. \( Q.E.D. \)

To derive such an equilibrium, we note that if none of the firms is constrained by its capacity, the standard Hotelling conditions (no infinitesimal deviations) then yield, as necessary conditions for a competitive equilibrium, a flat fee pricing structure for both firms with \( f^1 = f^2 = t \) and a profit of \( \frac{t}{2} \) for each firm. In this equilibrium, each firm’s capacity must be large enough to cover half of the market, i.e. \( K_i > \frac{d}{2} \). We now derive the sufficient conditions for this equilibrium.

The optimal deviation for any firm in this hypothesized equilibrium is to raise its prices such that the rival, charging the flat fee \( t \), becomes capacity-constrained. Without the rival being capacity-constrained, a firm’s best response to the rival’s charging the flat fee \( t \) is to charge the flat fee \( t \) itself. This implies that a firm may deviate only when the rival’s capacity is not large enough to cover the whole market, i.e. \( K_i < \bar{d} \). Otherwise, the optimal deviation can never make a firm better off. Consider the case where firm 2 unilaterally takes the optimal deviation, given that \( \frac{d}{2} < K_1 < \bar{d} \). In that case, firm 2’s prices will be such that the marginal light and heavy users to
firm 2 will all have zero surplus. As firm 1’s price in both segments is fixed at $t$, we need to specify a rationing rule to allocate firm 1’s capacity at that low price. For simplicity, we assume that firm 1’s capacity is allocated on the basis of location such that we always have $x^1_l = x^1_h = x^1$. This means $x^1 = \frac{K_1}{d}$. Here we can also use the efficient rationing rule. Such a rule will not qualitatively alter our conclusion, but will yield a far more complex cutoff point in capacity for the competitive equilibrium to be sustained.

Firm 2’s optimal deviation prices can be determined from $v_h d_h - p^2_h - t(1-x^1) = 0$ and $v_l d_l - p^2_l - t(1-x^1) = 0$. Then, the optimal deviation profit for firm 2 is given by $\alpha (1-x^1) p^2_h + (1-\alpha)(1-x^1) p^2_l$.

Let

$$\tilde{K}_c = \frac{d}{2} \left( \sqrt{\gamma^2 - 2 - (\bar{\gamma} - 2)} \right),$$

where $\bar{\gamma} = \alpha \gamma_h + (1-\alpha) \gamma_l > 2$ and $\frac{d}{2} < \tilde{K}_c < \bar{d}$. It is straightforward to show that as long as $K_1 > \tilde{K}_c$, firm 2’s optimal deviation profit is strictly less than $\frac{t}{2}$, its profit in the hypothesized equilibrium, so that firm 2 has no incentive to deviate. The same analysis also applies to firm 1.

$Q.E.D.$