Advertising in a Distribution Channel

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Technical Appendix: Proof of Proposition 3

Define $\Pi_{xy}^{[0,k]}$ to be the maximized overall channel profit after consumers located in $[0,k]$ have been exposed to advertising messages by firm $X$ that change their perception of product substitutability from $t$ to $t+\Delta t$ ($\Delta t > 0$). Define $\Pi_x^{[0,k]}$ and $\Pi_y^{[0,k]}$ analogously.

We begin with the effect of advertising on the joint profits of the retailer and firm $Y$. To determine $\Pi_y^{[0,k]}$ suppose that the profit maximizing price lies in $[V-\Delta(1+k), V-t]$. Then $p_y^m = V-(t+\Delta t)$ will be profit maximizing by $V > 2(t+\Delta t)$. Suppose $p_y^m > V - t$. Then $p_y^m = V - t$ will be profit maximizing by $V > 2(t+\Delta t) > 2t$. Hence, $\Pi_y^{[0,k]} = \max\{\Pi_y^{[0,1]}, \Pi_y^{[0,2]}\}$ where $\Pi_y^{[0,1]} = V-(t+\Delta t)$ and $\Pi_y^{[0,2]} = (V-t(1-k))(1-k)$.

We now consider the effect of advertising on total channel profits for different regions of $k$.

Small $k$: For $k \in [0, \frac{t}{2(\Delta t)}]$, $\Delta\Pi_{xy} = 0$ since for these $k$ the valuations of consumers in $[0,1/2]$ for product $X$ lie above $V-t/2$. Also $\Delta\Pi_y < 0$ for any $k > 0, \Delta t > 0$ by inspection of $\Pi_y$ above.

Medium $k$: For $k \in [\frac{t}{2(\Delta t)}, 1/2]$ define $\Pi_{xy}^{[0,k]} = V-(t+\Delta t)k$ as a lower bound of $\Pi_{xy}^{[0,k]}$. Simple calculations show that $\Pi_{xy}^{[0,k]} = \Pi_{xy} - \Pi_y^{[0,k]} - \Pi_y$ and $\Pi_{xy}^{[0,k]} = \Pi_{xy} - \Pi_y + \Pi_y^{[0,k]} - \Pi_y$ are monotonically decreasing in $k$ for $k \in [\frac{t}{2(\Delta t)}, 1/2]$. In addition one can easily show that $\Pi_{xy}^{[0,k]}(V_{xy} - \Pi_{y}^{[0,k]} - \Pi_{y} < 0$ and $\Pi_{xy}^{[0,k]} = \Pi_{xy} - \Pi_y^{[0,k]} + \Pi_y) < 0$ at $k = \frac{t}{2(\Delta t)}$. This shows that $|\Delta\Pi_{y}| > |\Delta\Pi_{xy}|$ for $k \in [\frac{t}{2(\Delta t)}, 1/2]$.

Large $k$: For $k \in (1/2, 1]$, $\Pi_y^{[0,1]} > \Pi_y^{[0,2]}$. Define $\Pi_{xy}^{[0,k]} = V-(t+\Delta t)/2$ as a lower bound of $\Pi_{xy}^{[0,k]}$ for these $k$. Clearly $|\Delta\Pi_{y}| = |\Delta t| > |\Pi_{xy}^{[0,k]} - \Pi_{xy}| = |t - \Delta t/2| > |\Delta\Pi_{xy}|$.

Hence, for any admissible $\Delta t$ and all $k$, $|\Delta\Pi_{y}| > |\Delta\Pi_{xy}|$ which jointly with $\Delta t > 0$ implies $\Delta\pi^*_x > 0$.

Now consider the effect of advertising on the profits of firm $Y$ by region of $k$.

Small $k$: For $k \in [0, t/(t+\Delta t)]$ the valuations of all consumers in $[0,k]$. 


exceed $V - t$. Recall that in the benchmark case $p^m_x = V - t$ maximizes the retailer's joint monopoly profits with firm $X$. Since all consumers' valuations still exceed $V - t$ after having been exposed to differentiating advertising, $p^m_x = V - t$ remains profit maximizing. Hence, $\Delta \Pi_x = 0$ for $k \in [0, t/(t+\Delta_t)]$.

We have also derived above that $\Delta \Pi_{xy} = 0$ for $k \in [0, \frac{t}{2(t+\Delta_t)}/]$. Since $\frac{t}{2(t+\Delta_t)} < \frac{t}{(t+\Delta_t)}$ for all admissible $t > 0$ and $\Delta_t > 0$, this shows that there are $k$ sufficiently small for which $\Delta \pi_y = \lambda_y(\Delta \Pi_{xy} - \Delta \Pi_x) = 0$.

**Intermediate $k$:** We know that $\Pi_x^{[0,k]} = V - t$ for $k \in [0, t/(t+\Delta_t)]$. Also, for any $k \in (\frac{t}{2(t+\Delta_t)}, 1]$ the retailer can either leave prices at the benchmark level and lose demand for consumers in $[\frac{t}{2(t+\Delta_t)}, k]$, or decrease prices while serving all consumers. In either case $\Delta \Pi_{xy} < 0$. Hence, for any $k \in (\frac{t}{2(t+\Delta_t)}, \frac{t}{(t+\Delta_t)})$ $\Delta \Pi_x = 0$, $\Delta \Pi_{xy} < 0$ and $\Delta \pi_y = \lambda_y(\Delta \Pi_{xy} - \Delta \Pi_x) < 0$.

**Large $k$:** For $k > 1/2$, a lower bound for $\Pi_x^{[0,k]}$ is given by $\Pi_x^{[0,k]} = (V - (t + \Delta_t)/2)$. We also know that $\Pi_x^{[0,k]} = V - (t + \Delta_t)k$ for $k \in [t/(t+\Delta_t), 1]$. Simple calculations using $\Pi_x^{[0,k]}$ show that for any $k \in (\frac{2t+\Delta_t}{2(t+\Delta_t)}, 1]$, $\Delta \Pi_x < \Pi_x^{[0,k]} - \Pi_{xy} \leq \Delta \Pi_{xy}$ and thus $\Delta \pi_y = \lambda_y(\Delta \Pi_{xy} - \Delta \Pi_x) > 0$.

Now consider the effect of advertising on the profits of the retailer.

Recall that for $\Delta t > 0$, $\Delta \pi_y^* = \lambda_x(\Delta \Pi_{xy} - \Delta \Pi_y) > 0$. Since $\lambda_x \geq 0$ this condition is equivalent to $\Delta \Pi_{xy} - \Delta \Pi_y > 0$. Now suppose that $\Delta \pi_y^* \geq 0$.

Since $\lambda_y \geq 0$ this condition is equivalent to $\Delta \Pi_{xy} - \Delta \Pi_x > 0$. We can write the change retailer’s profits as $\Delta \pi_y^* = \Delta \Pi_{xy} - \lambda_x(\Delta \Pi_{xy} - \Delta \Pi_y) - \lambda_y(\Delta \Pi_{xy} - \Delta \Pi_x)$. Since $\Pi_{xy} \leq 0$ for $\Delta t > 0$, $\Delta \pi_y^* < 0$.

Now assume that $\Delta \pi_y^* = \lambda_y(\Delta \Pi_{xy} - \Delta \Pi_x) < 0$. Since $\lambda_y \geq 0$ this assumption is equivalent to $\Delta \Pi_{xy} - \Delta \Pi_x < 0$. We can write the change retailer’s profits as $\Delta \pi_y^* = \Delta \Pi_{xy} - \lambda_x(\Delta \Pi_{xy} - \Delta \Pi_y) - \lambda_y(\Delta \Pi_{xy} - \Delta \Pi_x)$.

Since $\Delta \pi_y^* = \lambda_x(\Delta \Pi_{xy} - \Delta \Pi_y) > 0$, $\Delta \pi_y^* < 0$ if $\Delta \Pi_{xy} - \lambda_y(\Delta \Pi_{xy} - \Delta \Pi_x) \leq 0$.

We can rewrite this expression as $(1 - \lambda_y)\Delta \Pi_{xy} + \Delta \Pi_x \leq 0$. Since $\lambda_y \in [0, 1]$, $\Delta \Pi_{xy} \leq 0$, and $\Delta \Pi_x \leq 0$ (see above) this always holds. **Q.E.D.**