American Options under Stochastic Volatility:

Parameter Estimation and Pricing Efficiency

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Abstract

The stochastic volatility model of Heston (1993) is widely popular for its ability to capture many stylized facts of asset returns and for its resulting closed-form expressions for European option prices. However, its parameter estimation is challenging, and so is its application to the pricing of American options. In this paper, we present evidence that American option prices are insensitive to the accuracy of spot and long–term volatility estimates in the Heston (1993) model, for which drastically different parameter values can be derived. Our results derive from a new accurate pricing technique that we provide and which is based on a well-developed and efficient procedure for the constant volatility model of Black and Scholes. In addition, through an out–of–sample validation based on S&P 100 data, we also show that our method generates prices close to market values. In essence, our approach is predicated upon the classical Chernoff concentration bounds and the robustness of the Black-Scholes formula relative to misspecified stochastic volatility as shown by El Karoui et al. (1998).

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1 Introduction

In its most recent summary publication, the world's largest options market (CBOE) reports that its cumulative dollar volume for options was in excess of half a trillion dollars for the year 2015, with more than $74 billion traded in equities, for which 1.5 million average daily transactions were recorded (cf. CBOE (2015)). The importance of equity options markets is such that their interaction with markets for stocks has been the focus of extensive attention, particularly regarding direction and volume (see, e.g., Muravyev et al. (2013) and Hu (2014), and references therein.) For equity options, American–style exercise is predominant and volatility plays a central role in the appeal of early option exercise. However, the constant volatility assumption in the standard Black-Scholes model has long been challenged empirically. Among modeling alternatives, the literature on stochastic volatility is very large and is still growing, with diverse strands emphasizing different aspects such as forecast accuracy of realized volatility or fitting conditional distributions and option prices (cf. Christoffersen et al. (2010) for an overview.)

The mean–reverting stochastic volatility model of Heston (1993) has, in particular, found significant acceptance based on its ability to handle asymmetric fat tails and leverage as well as volatility clustering (see, e.g., Dragulescu and Yakovenko (2002).) One of the main reasons for the practical success of the Heston model is that it results in a nearly closed-form (analytic) pricing formula for European-style options. This in turn, when combined with the availability of option data, provides a convenient specification to estimate the unobservable volatility.

The Heston model is considered among the least parametrized, and a number of estimation techniques can be pursued, with values that may, however, differ substantially. They include the cross-section approach (cf., Bakshi et al. (1997), Bates (2000), and Huang (2004)), maximum likelihood based on filtering (Aït-Sahalia and Kimmel (2007), Christoffersen et al. (2010) , and Bates (2006)), and indirect inference (Gourieroux et al. (1993), AitSahlia et al. (2010b)).

Though stochastic volatility models do not account for certain stylized facts, which are usually captured through additional volatility components or jumps, they are considered to be the most essential building blocks upon which further refinements may be made. They capture enough complexity and yet are amenable to relatively manageable estimation and numerical pricing procedures. While the classical study of Bakshi et al. (1997) based on S&P 500 data provides empirical evidence that stochastic volatility delivers the most out–of-sample pricing improvement for European options, AitSahlia et al. (2010b) use S&P 100 data to show correspondingly that this aspect carries through to American-style exercise. Additionally, Broadie et al. (2000) employ non-parametric estimation on the same type of data to make the case for the need to incorporate stochastic volatility in American option pricing. Finally, Medvedev and Scaillet (2010) develop pricing formulas when both volatility and interest rates are stochastic. Through a numerical study, they show that both features have marked pricing effects relative to the Black–Scholes model, most significantly through mean–reverting stochastic volatility.

In their paper, Christoffersen et al. (2010) conduct an empirical validation of alternative stochastic volatility specifications comparable to Heston's in model parsimony and find that the most competitive is the one with linear, rather than Heston's square root, diffusion for variance. Specifically, it is the best fit for S&P 500 returns and their implied volatilities, and it also captures stylized facts in realized volatility. However, similar to all the alternatives compared in that paper, it results in pricing expressions for European options requiring significantly more intensive numerical evaluations than the Heston model.

When considering American options, however, pricing is a challenge even in the classical constant–volatility model of Black and Scholes (1973) for which a number of techniques have been developed and which can, in principle, be extended to account for stochastic volatility. They include free–boundary PDE methods, Monte–Carlo simulation, and stochastic mesh algorithms, among others (see e.g., Medvedev and Scaillet (2010) and Chockalingam and Muthuraman (2011) for reviews.)

Practitioners face the daunting task of evaluating several thousands of contracts simultaneously, where the vast majority can be exercised early, with trades increasingly occurring at less than one millisecond intervals. The presence of the early exercise feature exacerbates even more the pricing problem. In this paper, we follow a quasi–analytic approach whereby the price of an American option price is decomposed into that of the corresponding European option price and an early exercise premium that can be expressed in terms of the associated optimal early exercise boundary. When volatility is stochastic, this boundary is a surface depending on time and values of instantaneous volatility. Touzi (1994), for example, derives regularity results for the option price and shows that the optimal exercise boundary is a monotone function of the spot volatility. As mentioned above, different volatility estimates may be obtained, depending on the procedure used, which therefore call into question the accuracy of the resulting American option prices as well as the optimality of the associated early exercise strategy.

In this paper we show that this surface can be closely approximated based only on crude estimates of the long-term volatility of the mean–reverting Heston specification. Consequently, and similarly to the constant volatility specification, the optimal early exercise strategy only relies on a small set of critical values of the underlying asset that are deterministic and time–dependent. In so doing, we develop an efficient pricing algorithm for American options, which is especially important when dealing with large option books. More specifically, our method relies on the constant volatility model, for which fast and accurate techniques are well-developed, in order to determine the optimal exercise surface. The latter is then employed in the American option pricing decomposition formula for Heston's stochastic volatility model developed by Chiarella et al. (2010), which is also used in the regression–based technique of AitSahlia et al. (2010a). We will in particular exploit the accurate approximation technique of AitSahlia and

Lai (2001) to determine the constant volatility exercise boundary. Our approach is then compared to recent alternatives through systematic numerical scenarios. It is also evaluated with S&P 100 data to support it as a model that efficiently generates out–of–sample prices close to actuals.

The remainder of the paper is organized as follows: Section 2 summarizes the Heston model and presents the price decomposition formula for American options in this context. Section 3 expands on the method of constant volatility approximation approach. Section 4 compares it against recent numerical alternative techniques for a variety of numerical scenarios. In Section 5 it is evaluated empirically with market data. Section 6 concludes.

2 Heston's stochastic volatility model

Heston (1993) models both the stock and volatility as stochastic processes satisfying:

$$\begin{aligned} dS_t &= (\mu - q)S_t dt + \sqrt{v_t} S_t dW_1(t) \\ dv_t &= \kappa(\theta - v_t)dt + \eta\sqrt{v_t}\left(\rho dW_1(t) + \sqrt{1-\rho^2}dW_2(t)\right) \end{aligned} \tag{1}$$

where, under the original measure, μ and q are, respectively, the underlying mean return and dividend rates of the asset, $\sqrt{v_t}$ is its instantaneous volatility at time t, κ the mean reversion rate of the variance process $\{v_t\}$, θ its long-term mean, and η is the instantaneous volatility of volatility. In the above, S_t refers to the underlying asset (stock) price at time t and W_1 and W_2 are independent Wiener processes while ρ is the correlation between the innovations affecting the asset price and its volatility.

As long as $2\kappa\theta \geq \eta^2$ (cf. Feller (1951)), the values of v_t are guaranteed to remain positive for all t. Furthermore, negative estimates of ρ, which occurs commonly, capture the leverage effect whereby asset price declines are often accompanied with large volatilities.

Whereas the Black-Scholes model considers only one observable process, $\{S_t\}$, which can be used to estimate its sole unknown parameter required for pricing, namely the constant volatility of return, the Heston model introduces an additional process $\{v_t\}$ that is not observable and for which three parameters, κ, η, θ, must be estimated. An additional complication resulting from any stochastic volatility model is related to option pricing. Heston (1993) derives an analytic (nearly closed form) solution for European option but the corresponding problem of American option pricing is significantly more challenging and is not likely to result in any comparable formula.

Given that the number of sources of uncertainty exceeds available risky assets, the Heston model is incomplete and one way to select the risk–neutral pricing measure is through the use of a universal market price of volatility risk, denoted λ (cf., Fouque et al. (2000) for a discussion on market incompleteness in the context of stochastic volatility.) Under the risk neutral measure, the stochastic volatility model (1) becomes:

$$\begin{aligned} dS_t &= (r-q)S_t dt + \sqrt{v_t} S_t dZ_{1t} \\ dv_t &= \kappa^*(\theta^* - v_t)dt + \eta\sqrt{v_t}\left(\rho dZ_1(t) + \sqrt{1-\rho^2}dZ_2(t)\right) \end{aligned} \tag{2}$$

where $\kappa^* = \kappa + \lambda, \theta^* = \kappa\theta/(\kappa+\lambda)$, and Z_1 and Z_2 are two independent standard Brownian motions defined over an associated filtered probability space.

The estimation of the parameters for the specification above is a challenging problem. On the one hand, one might want to rely on the observed values of the underlying asset in order to estimate its structural parameters ($\mu, q, \kappa, \theta, \eta$ and ρ) under the original (physical) probability measure. On the other, one could use options data in order to estimate, under the risk-neutral measure, the market price of volatility risk (λ) and the starred versions of the structural parameters. However, severe discrepancies could occur. For example, Bakshi et al. (1997) show a direct estimation of ρ to be -0.23 all the while the estimate implied by options data is $-.76$. This issue of calibration between objective and pricing measures is prevalent in many specifications as discussed at length in Chernov and Ghysels (2000). As mentioned earlier in our introduction, a variety of approaches have been developed, often leading to significant differences in the estimated values. One may then wish to asses the impact of these discrepancies on option prices. Our objective in this paper is to assess this impact on American option prices for which estimates have been obtained via a two-step procedure based on indirect inference (AitSahlia et al. (2010b)), specifically those for spot and long-term (equilibrium) volatilities.

3 Decomposition formula for American put under stochastic volatility

If at time t the stock price is S and the volatility of its return is v, then the American put price $P_A(S, v, t)$ satisfies

$$\begin{aligned} rP_A =& \frac{\partial P_A}{\partial t} + (r-q)S\frac{\partial P_A}{\partial S} + \frac{1}{2}vS^2\frac{\partial^2 P_A}{\partial S^2} \\ &+ \kappa^*\left(\theta^* - v\right)\frac{\partial P_A}{\partial v} + \frac{1}{2}\eta^2 v\frac{\partial^2 P_A}{\partial v^2} + \rho\eta vS\frac{\partial^2 P_A}{\partial S\partial v} \end{aligned} \tag{3}$$

in the no–exercise region $\{(S, v, \tau) : S > b(v, t)\}$, where $b(v, t)$ is the optimal exercise boundary that is determined jointly with P_A satisfying (3) above and such that the following also hold:

$$
\begin{aligned}
P_A(S,v,T) &= (K-S)^+ && (4)\\
P_A(b(v,t),v,t) &= K-b(v,t) && (5)\\
\lim_{S\to b(v,t)} \frac{\partial P_A}{\partial S}(S,v,t) &= -1, && (6)
\end{aligned}
$$

with the latter two conditions capturing the optimality of $b(v,t)$ (smooth fit.)

In the Black-Scholes model, volatility is constant and the terms on the second line of (3) vanish. In this case, American option prices and their hedging parameters can be determined very efficiently and very accurately thanks to the classic decomposition formula expressing the American option price as that of the corresponding European option augmented by an early exercise premium (Carr et al. (1992), Kim (1990), Jacka (1991).). The latter involves the early exercise boundary, which is shown in an exhaustive and systematic study by AitSahlia and Lai (1999) to be approximately piecewise linear (on the log-price scale), requiring only a few knots to yield very accurate prices (AitSahlia and Lai (2001).)

Chiarella et al. (2010) obtain a decomposition formula for the price of an American call option in the context of Heston's stochastic volatility model (see their Proposition 10, p.293.) Though the put-call parity does not apply to American options (except for non–dividend paying calls), the adaptation of the Chiarella et al. (2010) approach to put options is relatively straightforward. In the interest of completeness, and in order to display the decomposition formula in this case, we first need to recall some functions from Chiarella et al. (2010). Namely,

$$
\begin{aligned}
\Lambda(\zeta) &\equiv i\zeta-\zeta^2 && (7)\\
\Theta(\zeta) &\equiv \beta+\eta\rho i\zeta && (8)\\
\Omega(\zeta) &\equiv \sqrt{\Theta^2(\zeta)-\eta^2\Lambda(\zeta)}, && (9)
\end{aligned}
$$

for a complex argument ζ, where $\beta = \kappa + \lambda$. Then the American put price $P_A(S, v, \tau)$, when the spot volatility is v and τ units of time are left to maturity, is given by

$$\begin{aligned} P_A(S,v,\tau) = {} & Ke^{-r\tau}\bar{P}_2(S,v,\tau;K,0) - Se^{-q\tau}\bar{P}_1(S,v,\tau;K,0) \\ & + \int_0^\tau \int_0^\infty rKe^{-r(\tau-\xi)}\bar{P}_2(S,v,\tau-\xi;w,b(w,\xi))dwd\xi \\ & - \int_0^\tau \int_0^\infty qSe^{-q(\tau-\xi)}\bar{P}_1(S,v,\tau-\xi;w,b(w,\xi))dwd\xi, \end{aligned} \tag{10}$$

where, for $j = 1, 2$,

$$\bar{P}_j(S,v,\tau;\alpha,\psi) \equiv \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{e^{-i\phi\ln\alpha}}{i\phi} f_j(S,v,\tau;\phi,\psi)\right)d\phi \tag{11}$$

and

$$\begin{aligned} f_j(S,v,\tau;\phi,\psi) & \equiv \exp\left\{B_j(\phi,\psi,\tau) + D_j(\phi,\psi,\tau)v + i\phi\ln S\right\} \\ B_j(\phi,\psi,\tau) & \equiv i\phi(r-q)\tau + \frac{\alpha}{\eta^2}\left\{(\Theta_j+\Omega_j)\tau - 2\ln\left(\frac{1-Q_je^{\Omega_j\tau}}{1-Q_j}\right)\right\} \\ D_j(\phi,\psi,\tau) & \equiv i\psi + \frac{(\Theta_j - \eta^2 i\psi + \Omega_j)}{\eta^2}\left(\frac{1-e^{\Omega_j\tau}}{1-Q_je^{\Omega_j\tau}}\right) \end{aligned} \tag{12}$$

with $Q_j \equiv (\Theta_j - \eta^2 i\psi + \Omega_j) / (\Theta_j - \eta^2 i\psi - \Omega_j)$, $\Theta_1 \equiv \Theta(i-\phi)$, $\Theta_2 \equiv \Theta(-\phi)$, $\Omega_1 \equiv \Omega(i-\phi)$, an $\Omega_2 \equiv \Omega(-\phi)$. We should note here that the first line in the expression (10) above is the price of the corresponding European put and the remaining two lines capture the early exercise premium, which requires the determination of the early exercise boundary $b(v,t)$, where v is the instantaneous variance at time t. This boundary is a surface separating in the (v,t)–space the optimal exercise region, where the put value is its payoff, and the continuation region, where the put option value satisfies (3). Given the decomposition formula (10), $b(v,t)$

solves the integral equation

$$
\begin{aligned}
K - b(v,\tau) = {} & Ke^{-r\tau}\bar{P}_2(S,v,\tau;K,0) - b(v,\tau)e^{-q\tau}\bar{P}_1(S,v,\tau;K,0) \\
& + \int_0^\tau \int_0^\infty rKe^{-r(\tau-\xi)}\bar{P}_2(b(v,\tau),v,\tau-\xi;w,b(w,\xi))dwd\xi \\
& - \int_0^\tau \int_0^\infty qb(v,\tau)e^{-q(\tau-\xi)}\bar{P}_1(b(v,\tau),v,\tau-\xi;w,b(w,\xi))dwd\xi,
\end{aligned}
\tag{13}
$$

Given expression (11), the above equation requires the (numerical) evaluation of triple integrals, which can be very burdensome. Based on the empirical evidence in Broadie et al. (2000) suggesting an approximate linear relationship between $\ln b(v,t)$ and v, namely

$$
\ln b(v,t) \approx b_0(t) + vb_1(t), \tag{14}
$$

where $b_0(t)$ and $b_1(t)$ are deterministic functions of t, Adolfsson et al. (2013) obtain a decomposition formula for the price of an American call option in the context of Heston's stochastic volatility model. As a result, the manage to reduce the integration dimensionality to two and their approach still requires solving numerically for the roots of two-dimensional non–linear systems, which are prone to numerical instability. We propose to go one step further in dimension reduction by solving the integral equation by approximating the optimal exercise surface with a volatility invariant one; i.e., by setting $b_1 \equiv 0$. This is a clearly a faster method than that of Adolfsson et al. (2013). The focus of current alternative techniques on puts is behind our adaptation of Adolfsson et al. (2013) to puts. With the linear approximation of log of the surface, we have for the American put:

$$\begin{aligned} P_A(S,v,\tau) = Ke^{-r\tau}\bar{P}_2(S,v,\tau;K,0) &- Se^{-q\tau}\bar{P}_1(S,v,\tau;K,0) \\ &+ \int_0^\tau rKe^{-r(\tau-\xi)}\bar{P}_2(S,v,\tau-\xi;e^{b_0(\xi)},-b_1(\xi))d\xi \\ &- \int_0^\tau qSe^{-q(\tau-\xi)}\bar{P}_1(S,v,\tau-\xi;e^{b_0(\xi)},-b_1(\xi))d\xi, \end{aligned} \tag{15}$$

To complete this expression, the early exercise premium (integral terms on the right-hand-side of (4) above) we need the deterministic function $b_0(t)$ and $b_1(t)$, which are obtainable through the matching condition upon exercise:

$$P_A\left(b(v,\tau),v,\tau\right) = K - b(v,\tau).$$

This is a significantly complex two-dimensional integral equation and we introduce next an approximation technique to reduce it to a more manageable one-dimensional integral equation, which results from our approximation based on a variance-invariant, but time–dependent, optimal exercise surface.

4 Early exercise boundary approximation

In this section we show how we can adapt to the Heston stochastic volatility model a technique initially developed for the classical constant–volatility Black-Scholes model. In the latter, the underlying asset price process $\{S_t\}$ is assumed to follow a geometric Brownian motion under the risk-neutral measure:

$$dS_t = (r-q)S_t dt + \sigma S_t dW_t,$$

where σ is its volatility of return, q its dividend rate, r the market risk-free rate, and W_t a standard Brownian motion. Given the comparison context for stochastic volatility in forthcoming sections, we focus here on an American put written on this underlying. Its maturity is labeled T while its strike is K. Then at time t, for a spot price S, the American put option price $P(S,t)$ can be decomposed as:

$$\begin{aligned} P(S,t) =& P_E(S,t) + rK \int_t^T e^{-r(\tau-t)} N(-d_2(S; B_\tau, \tau - t)) \\ & - qS \int_t^T e^{-q(\tau-t)} N(-d_1(S; B_\tau, \tau - t)) d\tau \end{aligned} \tag{16}$$

where $d_1(x;y,\tau) = \left(ln(x/y) + (r - q + \frac{1}{2}\sigma^2)\tau\right)/\sigma\sqrt{t}$, $d_2(x;y,\tau) = d_1(x;y,\tau) - \sigma\sqrt{t}$, $P_E(S,t)$ is the price of the corresponding European option, and $N(x)$ is the cumulative standard normal distribution function (cf. Kim (1990), Jacka (1991) and Carr et al. (1992).) As the boundary $\{B_t\}$ is unknown in this expression, we use the matching condition upon exercise $P(B_t,t) = K - B_t$ to get an integral equation for this boundary, namely:

$$\begin{aligned} K - B_t = & P_E(B_t, T-t) + \\ & \int_t^T \left[rKe^{-r(\tau-t)} N(-d_2(B_t, B_\tau, \tau - t)) - qB_t e^{-q(\tau-t)} N(-d_1(B_t, B_\tau, \tau - t)) \right] d\tau \end{aligned} \tag{17}$$

This integral equation has been the focus of much research effort. Huang et al. (1996) approximate the early exercise boundary with piecewise constant functions, using only a few pieces, typically 3 or 4. Ju (1998) followed the same general idea, using piecewise exponential approximations instead. By showing that large classes of American option pricing problems can be reduced to canonical pricing problems indexed by one or two parameters, AitSahlia and Lai (1999) conduct an extensive study showing that the early exercise boundary for the constant volatility model is indeed well-approximated by a piecewise exponential function. Furthermore, they make use of spline approximations in AitSahlia and Lai (2001) to efficiently determine this piecewise exponential function. Their approach in fact improves upon that of Ju (1998) as it makes use of root-finding algorithms in one dimension instead of two for the latter, which are prone to stability issues. In addition, their boundary approximation is the only one that is continuous, which is in conformity with its theoretical characterization. We therefore use the same method as in AitSahlia and Lai (2001) to approximate the constant volatility boundary to generate the stochastic volatility surface in the next section.

Following AitSahlia and Lai (1999), we perform in (2) the change of variables:

$$s = \sigma^2(t - T), \qquad z = \log(S/K) - (\rho - \alpha\rho - \frac{1}{2})s,$$

where $\rho = r/\sigma^2$ and $\alpha = q/r$, to obtain the following integral formula for the boundary ($\bar{z}(s)$) in the canonical form

$$\begin{aligned} 1 - e^{\bar{z}(s)+(\rho-\alpha\rho-\frac{1}{2})s} =& e^{\rho s}\left[N\left(\frac{-\bar{z}(s)}{\sqrt{-s}}\right) - e^{\bar{z}(s)-\frac{1}{2}s}N\left(\frac{-\bar{z}(s)}{\sqrt{-s}} - \sqrt{-s}\right)\right] + \rho e^{\rho s} \\ & \int_s^0 \left[e^{-\rho u}N\left(\frac{\bar{z}(u)-\bar{z}(s)}{\sqrt{u-s}}\right) - \alpha e^{-\alpha\rho u+\frac{1}{2}s+\bar{z}(s)}N\left(\frac{\bar{z}(u)-\bar{z}(s)}{\sqrt{u-s}} - \sqrt{u-s}\right)\right] du \end{aligned} \tag{18}$$

An advantage of this change of variables is the significant reduction of the interval over which to conduct computational procedures (a fraction σ^2 of the original.) Dividing the interval $[s, 0]$ into m subintervals such that $s = s_m < \ldots < s_0 = 0$ and proceeding in the same way as in AitSahlia and Lai (2001), the boundary $\bar{z}()$ is solved recursively starting from

$$z(0) = \begin{cases} 0 & \text{if } 0 \leq \alpha \leq 1 \\ -\ln \alpha & \text{if } \alpha > 1 \end{cases} \tag{19}$$

As the approximating boundary is piece-wise linear, each intercept is determined by the previous piece and thus only the corresponding slope needs to be determined as root of a non-linear equation as explained next.

With $\bar{z}_j = \bar{z}(s_j)$ and $\tau_j = s_j - s_m$, once $\bar{z}_0, \ldots, \bar{z}_{m-1}$ are determined, $\bar{z}_m$ can be determined by solving the following equation for z

$$\begin{aligned} 1 - e^{z+(\rho-\alpha\rho-\frac{1}{2})s_m} =& e^{\rho s_m}\left[N\left(\frac{-z}{\sqrt{-s_m}}\right) - e^{z-\frac{1}{2}s_m} N\left(\frac{-z}{\sqrt{-s_m}} - \sqrt{-s_m}\right)\right] \\ &+ 1 - e^{\rho s_m} - e^{z+(\rho-\alpha\rho-\frac{1}{2})s_m}(1 - e^{\rho\alpha s_m}) \\ &+ e^{-\rho\tau_{m-1}} N(b(z)\tau_{m-1}^{1/2}) - \frac{1}{2} - \frac{b(z)}{a(z)} N(a(z)\tau_{m-1}^{1/2} - 0.5) \\ &- \sum_{i=1}^{m-1} A_i(z) + e^{z+(\rho-\alpha\rho-\frac{1}{2})s_m}\left[\frac{\tilde{b}(z)}{\tilde{a}(z)}[N(\tilde{a}(z)\tau_{m-1}^{1/2}) - 0.5]\right] \\ &+ \frac{1}{2} - e^{-\alpha\rho s_{m-1}} N(\tilde{b}(z)\tau_{m-1}^{1/2}) + \sum_{i=1}^{m-1} \tilde{A}_i(z) \end{aligned} \tag{20}$$

where $A_i(z)$ and $\tilde{A}_i(z)$ are given by the RHS of equations (9) and (10) in AitSahlia and Lai (2001) and

$$\begin{aligned} b(z) &= \frac{z - z_{m-1}}{s_{m-1} - s_m} \\ a(z) &= [b^2(z) + 2\rho]^{1/2} \\ \tilde{b}(z) &= b(z) + 1 \\ \tilde{a}(z) &= [\tilde{b}^2(z) + 2\alpha\rho]^{1/2} \end{aligned}$$

To solve equation (20) we use the bisection method, for which we use the lower and upper bounds for the put option boundary (AitSahlia and Lai (1999)) as starting points:

$$\bar{z}_u(s) = -[\rho(1-\alpha) - 0.5]s - ln(\alpha)^+, \tag{21}$$

$$\bar{z}_l(s) = -[\rho(1-\alpha) - 0.5]s - ln(\bar{\beta}/(\bar{\beta}-1)) \tag{22}$$

where $\bar{\beta} = -\left[\rho(1-\alpha) - \frac{1}{2}\right] + \left\{\left[\rho(1-\alpha) - \frac{1}{2}\right]^2 + 2\rho\right\}^{1/2}$

The obtained boundary values $\bar{z}(0), \bar{z}(1), \ldots, \bar{z}(m)$ are then converted from the canonical form back to the standard form by using

$$B_t = Ke^{z(s)+\left(\rho-\alpha\rho-\frac{1}{2}\right)s} \tag{23}$$

where $s = \sigma^2(t-T)$.

4.1 Constant–volatility surface approximation

Driven by practical implementation considerations and motivated by the fact that the (one–dimensional) optimal exercise boundary in the Black–Soholes model can be approximated rather crudely (e.g., piecewise constant) and still yield fairly accurate prices, we propose a similar simple, yet accurate approach when dealing with stochastic volatility. The decomposition formula for the American put under stochastic volatility expressed in (15) assumes that the early exercise surface $b(v,t)$ is of the form $b(v,t) = \exp\{b_0(t) + vb_1(t)\}$. Setting $b_1 \equiv 0$, the corresponding integral equation is unidimensional in the same vein as (17), namely a deterministic function $\exp\{b_0(t\})$ that plays the role of B_t therein. However, the quantities defined in (15) require a value for the instantaneous volatility v, which we vary in our numerical and empirical validations (Sections 5 and 6, respectively.) In what follows, we refer to our implementation as CV-Decomp (Constant Volatility-Decomposition) method. It should be noted that neither the approximation (14) above due to Broadie et al. (2000) nor its restriction to the affine form that we propose will lead to an option price satisfying the associated free–boundary problem (3) – (6). However, what we argue is that, similarly to the original Black–Scholes setup, using a seemingly non–optimal exercise boundary does not prevent the generation of accurate options prices. Clearly, this type of approximation does not fit a standard setup where convergence can be assessed theoretically (cf. Huang et al. (1996), Ju (1998), and AitSahlia and Lai (2001).) Therefore, we focus on the practical impact of our approach by assessing its accuracy via (i) a comparison with alternative PDE–based approaches, and (ii) an out–of–sample empirical validation.

[*This section to be further expanded with theoretical arguments regarding the validity of our approximation approach, based on the Chernoff concentration bound and approximations to the non–central* χ^2 *distribution.*]

5 Comparison against PDE-based approaches

We assess the efficiency of our approach against those of recent PDE-based numerical methods considered in Ikonen and Toivanen (2007) and Chockalingam and Muthuraman (2011). Ikonen and Toivanen (2007) develop a componentwise splitting method to compute American put option prices and compare it with those of Clarke and Parrott (1996), Zvan et al. (1998) and Oosterlee (2003). Chockalingam and Muthuraman (2011) develop a method that transforms the free–boundary problem arising in American option pricing into a sequence of fixed-boundary problems involving European–type options. We therefore use the parameter values used in both papers to assess the accuracy and computational efficiency of our approach against all these alternatives. In essence, this is one way to gauge how well we solve the optimal stopping problem associated with the pricing of an American option with our approach. We should note, in passing, that we did not compare with Beliaeva and Nawalkha (2010) as their method is based on the trinomial tree, which is significantly slower than any quasi-analytic approach such as ours.

Ikonen and Toivanen (2007) compared the speed and accuracy of five numerical methods to price American puts in the Heston model: their component-wise splitting technique, the projected SOR, the projected multi grid method of Clarke and Parrott (1996), the operator splitting method of Oosterlee (2003), and the penalty method of Zvan et al. (1998). Their study indicates that their approach is the fastest with an accuracy comparable to the others.

Ikonen and Toivanen (2007) computed put prices with strike $K = 10$, a maturity of $T = .25$, and spot prices $S_0 = 8, 9, 10, 11$ and 12, for two spot volatilities: $v_0 = 0.0625$ and $v_0 = 0.25$. They report the time for a particular grid size (1280; 512; 258) to be 56.82 seconds on 3.40 GHz Intel Xeon PC. For the Heston model, the parameters used are: $\kappa = 5.00, \theta = 0.16, \eta = 0.9, \rho = 0.1, \lambda = 0, r = 0.1, q = 0.0$. In anticipation of the more systematic comparison we present next, we calculated the early-exercise boundary by assuming σ to be equal to one of two different values: $\sigma^2 = \theta$ and $\theta + \sqrt{\theta}\eta$. In reference to the next set of numerical tests, we could not use $\sigma^2 = \theta - \sqrt{\theta}\eta$ as it was negative. For our implementation, we computed the put option prices with the same parameters, considering constant volatility boundaries with 3, 5, 10, 25 and 50 pieces for the two values of σ above. Table 1 reports the mean prices for these five runs. The corresponding standard deviations are given in parentheses. Their values indicate that there is little variation between the different runs. Ikonen and Toivanen (2007) list their best time as 56.82 seconds on a 3.40 GHz Intel Xeon PC. Chockalingam and Muthuraman (2011) implement their technique on a 2.8 GHz Intel Xeon Mac Pro and report comparable accuracy to Ikonen and Toivanen (2007) while doubling the computational speed. However, ours is even much faster, for comparable accuracy. For example, the

Table 1: **American put price comparison**: strike $K = 10$, maturity $T = .25$, spot price S_0 and two spot volatility values v_0. Parameters for Heston's stochastic volatility model: $\kappa = 5.00, \theta = 0.16, \eta = 0.9, \rho = 0.1, \lambda = 0, r = 0.1, q = 0.0$. CV-Decomp refers to our constant volatility – decomposition formula approach with $\sigma^2 = \theta$. Entries for this approach are the average of the option prices obtained with 3–, 5–, 10–, 25– and 50-piece approximate exercise boundaries (with their standard deviations in parentheses). Other approaches are taken from Ikonen and Toivanen (2007) and Chockalingam and Muthuraman (2011). The values for their finite–difference schemes are averaged for the four grid sizes they considered and standard deviations are provided in parentheses. Our longest computation time (with the 50–piece boundary took 0.05 seconds on 2.00 GHz Intel Core2 PC while the best time for Ikonen and Toivanen (2007) was reported as 56.82 seconds on a 3.40 GHz Intel Xeon PC. Chockalingam and Muthuraman (2011) run theirs on a 2.8 GHz Intel Xeon Mac Pro and report a computational time that is half of that of Ikonen and Toivanen (2007).

Method	v_0	S_0				
		8	9	10	11	12
CV-Decomp	0.0625	1.9486	1.0832	0.5189	0.2212	0.0891
		(0.0023)	(0.0021)	(0.0043)	(0.0016)	(0.0002)
	0.25	2.0867	1.3345	0.8001	0.4568	0.2528
		(0.0042)	(0.0011)	(0.0020)	(0.0013)	(0.0004)
Ikonen and Toivanen (2007)	0.0625	2.0001	1.1046	0.5129	0.2099	0.0820
		(0.0002)	(0.0031)	(0.0094)	(0.0045)	(0.0006)
	0.25	2.0747	1.3257	0.7858	0.4401	0.2385
		(0.0048)	(0.0110)	(0.0143)	(0.0114)	(0.0059)
Chockalingam and Muthuraman (2011)	0.0625	2.0000	1.1030	0.5120	0.2101	0.0823
		(0.0000)	(0.0054)	(0.0105)	(0.0041)	(0.0010)
	0.25	2.0752	1.3270	0.7880	0.4420	0.2393
		(0.0041)	(0.0088)	(0.0108)	(0.0085)	(0.0046)
Clarke and Parrott (1996)	0.0625	2.0000	1.1080	0.5316	0.2261	0.0907
	0.25	2.0733	1.13290	0.7992	0.4536	0.2502
Zvan et al. (1998)]	0.0625	2.0000	1.1076	0.5202	0.2138	0.0821
	0.25	2.0784	1.3337	0.7961	0.4483	0.2428
Oosterlee (2003)	0.0625	2.0000	1.1070	0.5170	0.2120	0.0815
	0.25	2.0794	1.3340	0.7962	0.4490	0.2431

maximum time taken (with the 50-piece boundary) is 0.05 seconds on a modest 2.00 GHz Intel Core2 PC.

6 Empirical Validation

AitSahlia et al. (2010b) use the least–squares Monte Carlo (LSM) method of Longstaff Schwartz algorithm (Longstaff and Schwartz, 2001) to price American options under constant and stochastic volatility using actual market data. They find that Heston's stochastic volatility model improves the pricing and hedging performance over the constant volatility model, thus extending earlier results obtained by Bakshi et al. (1997) for European options. In this section, we use the same data in order to validate our approach which combines a constant–volatility model to determine the early exercise boundary with a stochastic–volatility model to determine prices. We rely on the same data in order to compare directly with a method (LSM) that makes full use of the Heston specification in AitSahlia et al. (2010b), in contrast to the partial reliance in the present paper on constant volatility.

6.1 Data description and parameter estimates

The data consist of the daily closing prices of S&P 100 index options, which are of both European and American exercise styles. The entire sample period for estimation and performance evaluation is January 1, 2002 to April 28, 2006. The reported price of any given option is the average of its bid and ask prices. In AitSahlia et al. (2010b) we used data on European style S& P 100 index options to estimate the parameters in Heston's model. The same estimates are used here and are reproduced in Table 2.

The calibrated Heston model is used for performance evaluation of American put option prices using CV-Decomp approach with 3 boundary pieces. The out–of–sample data are from

January 1, 2006 to April 28, 2006. Applying the exclusionary criteria of AitSahlia et al. (2010b), leaves $8,177$ put options for 81 days in the sample period (an average of 101 options per day.). As shown through the numerical evaluation in the previous sections, our approach is robust with respect to volatility estimates. Therefore, we use long–term mean $\sqrt{\theta}$ and spot volatility $\hat{v}$ calculated in AitSahlia et al. (2010b) as potential values for the constant volatility σ and analyze the performance of the proposed model for these estimates.

6.2 Classification of options

We classify options according to maturity(T, in days) and moneyness (x), which for a put option is defined as $x = K/S - 1$, where S is the current spot and K the strike. Different option groups are defined below. In our analysis, short-term options are those for which ($T < 45$); mid-term options have ($45 \leq T < 90$) and those with ($T \geq 90$) are labeled long–term options. With respect to moneyness, we classify options with $x > 0.05$ as deep-in-the-money (DITM), those with $x \in (0.02, 0.05)$ as in-the-money (ITM) options, those with $x \in (-0.02, 0.02)$ as at-the-money (ATM) options, those with $x \in (-0.05, -0.02)$ as out-of-the-money (OTM) options, and those with $x > -0.05$ as deep-out-of-the-money (DOTM) options.

Table 2: Parameter estimates of Heston's model

Parameter	Estimate
κ	2.65
θ	0.029
η	0.154
$\hat{\rho}$	-0.487
$\hat{v}$	0.0349
$\hat{\lambda}$	2.14

6.3 Measures of performance

We compare the LSM method in AitSahlia et al. (2010b) and CV-Decomp approach according to out-of-sample mean relative-pricing errors and absolute-pricing errors, following the option classification described above. For the ith option in a given group of n, we let C_i^M and C_i denote the observed market and model prices, respectively. Then the mean relative pricing error for the group is defined as

$$MRE = \frac{1}{n}\sum_{i=1}^{n}\frac{(C_i^M - C_i)}{C_i}$$

and its mean absolute relative pricing error as:

$$MAE = \frac{1}{n}\sum_{i=1}^{n}\frac{|C_i^M - C_i|}{C_i}.$$

MRE is an indicator of the pricing bias whereas MAE evaluates the magnitude of mispricing. The results in tables 3 through 6 are obtained by computing these averages for each moneyness-maturity group. Since we have a choice in setting the initial volatility, namely the spot volatility or the long-term mean θ, we chose to consider both as displayed in these tables.

6.4 Empirical Analysis

We use in this paper the same classification scheme and metrics for performance evaluation as in AitSahlia et al. (2010b). The results in Tables 3 and 4 are obtained by computing the average of percentage pricing errors for each moneyness-maturity group when constant and initial volatility are set to spot volatility and long term average, respectively. The results in Tables 5 and 6 are obtained by computing the average of absolute pricing errors for each moneyness-maturity group when constant and initial volatility are set to spot volatility and long term average, respectively. LSM represents the results from LSM–based stochastic volatility model in AitSahlia et al. (2010b) together with their reported standard error values. 3CV-Decomp refers to the results obtained from our proposed approach with an early exercise boundary approximated by a three-piece log–linear function, which is then input in the decomposition formula for the SV model.

Table 3: Out-of-sample relative pricing errors for initial and constant volatility approximation σ set at spot volatility

		Maturity		
Money-ness	Model	Short-Term	Mid-Term	Long-Term
DITM	LSM	0.002	0.019	0.019
		(0.001)	(0.002)	(0.003)
	3CV-Decomp	-0.003	0.019	0.010
ITM	LSM	0.133	0.241	0.228
		(0.006)	(0.006)	(0.011)
	3CV-Decomp	0.136	0.250	0.237
ATM	LSM	0.991	0.699	0.479
		(0.032)	(0.014)	(0.016)
	3CV-Decomp	1.045	0.702	0.473
OTM	LSM	2.049	1.217	0.66
		(0.049)	(0.026)	(0.025)
	3CV-Decomp	2.146	1.161	0.614
DOTM	LSM	1.411	0.969	0.411
		(0.054)	(0.025)	(0.029)
	3CV-Decomp	-0.391	-0.116	-0.323

LSM refers to results based on the least-squares algorithm of Longstaff and Schwartz (2001), with associated standard errors, as reported in AitSahlia et al. (2010b); 3-CV-Decomp refers to our approach where the early exercise boundary is approximated by a three-piece linear functions. The reported absolute pricing error is the absolute value of market price minus the model price divided by the market price for each moneyness-maturity category.

Table 4: Out-of-sample relative pricing errors for initial and constant volatility approximation σ set at long term average

		Maturity		
Money-ness	Model	Short-Term	Mid-Term	Long-Term
DITM	LSM	-0.009	0.000	0.003
		(0.001)	(0.001)	(0.002)
	3CV-Decomp	-0.014	-0.012	-0.009
ITM	LSM	0.072	0.146	0.139
		(0.004)	(0.004)	(0.008)
	3CV-Decomp	0.004	0.045	0.036
ATM	LSM	0.697	0.449	0.288
		(0.021)	(0.008)	(0.009)
	3CV-Decomp	0.224	0.121	0.031
OTM	LSM	1.149	0.688	0.347
		(0.027)	(0.014)	(0.014)
	3CV-Decomp	-0.062	-0.013	-0.106
DOTM	LSM	0.484	0.226	-0.071
		(0.034)	(0.016)	(0.019)
	3CV-Decomp	-0.916	-0.808	-0.795

LSM refers to results based on the least-squares algorithm of Longstaff and Schwartz (2001), with associated standard errors, as reported in AitSahlia et al. (2010b); 3-CV-Decomp refers to our approach where the early exercise boundary is approximated by a three-piece linear functions. The reported absolute pricing error is the absolute value of market price minus the model price divided by the market price for each moneyness-maturity category.

Table 5: Out-of-sample absolute relative pricing errors for initial and constant volatility approximation set at spot volatility

		Maturity		
Money-ness	Model	Short-Term	Mid-Term	Long-Term
DITM	LSM	0.019	0.028	0.029
		(0.001)	(0.002)	(0.003)
	3CV-Decomp	0.016	0.035	0.038
ITM	LSM	0.136	0.241	0.228
		(0.006)	(0.006)	(0.011)
	3CV-Decomp	0.139	0.250	0.238
ATM	LSM	0.991	0.699	0.479
		(0.032)	(0.014)	(0.016)
	3CV-Decomp	1.045	0.702	0.473
OTM	LSM	2.048	1.217	0.660
		(0.049)	(0.026)	(0.025)
	3CV-Decomp	2.146	1.161	0.614
DOTM	LSM	1.419	1.002	0.514
		(0.053)	(0.023)	(0.022)
	3CV-Decomp	0.911	0.767	0.640

LSM refers to results based on the least-squares algorithm of Longstaff and Schwartz (2001), with associated standard errors, as reported in AitSahlia et al. (2010b); 3-CV-Decomp refers to our approach where the early exercise boundary is approximated by a three-piece linear functions. The reported absolute pricing error is the absolute value of market price minus the model price divided by the market price for each moneyness-maturity category.

Table 6: Out-of-sample absolute pricing errors for initial and constant volatility approximation set at long–term average

		Maturity		
Money-ness	Model	Short-Term	Mid-Term	Long-Term
DITM	LSM	0.015	0.017	0.019
		(0.000)	(0.001)	(0.001)
	3CV-Decomp	0.015	0.015	0.013
ITM	LSM	0.079	0.147	0.140
		(0.004)	(0.004)	(0.007)
	3CV-Decomp	0.034	0.048	0.039
ATM	LSM	0.647	0.449	0.288
		(0.022)	(0.008)	(0.009)
	3CV-Decomp	0.229	0.125	0.049
OTM	LSM	1.149	0.689	0.347
		(0.027)	(0.014)	(0.014)
	3CV-Decomp	0.252	0.109	0.131
DOTM	LSM	0.571	0.394	0.281
		(0.016)	(0.007)	(0.011)
	3CV-Decomp	0.916	0.808	0.795

LSM refers to results based on the least-squares algorithm of Longstaff and Schwartz (2001), with associated standard errors, as reported in AitSahlia et al. (2010b); 3-CV-Decomp refers to our approach where the early exercise boundary is approximated by a three-piece linear functions. The reported absolute pricing error is the absolute value of market price minus the model price divided by the market price for each moneyness-maturity category.

Let us first consider the percentage and absolute pricing errors listed in tables 3 and 5 which correspond to prices obtained when initial volatility is set to spot volatility. Considering both pricing errors, we observe that our approximation approach performs better for DOTM, ITM, ATM and OTM moneyness groups and for mid-term and long-term maturities. For the deep-in-the-money options, our model's pricing accuracy is comparable with that of LSM approach. The errors from approximation are within 2% difference. It also shows that for deep in the money options the models have no particular bias towards overpricing or underpricing of options and the absolute pricing errors for this moneyness group have considerably lower errors than the CV model. However, for deep out of the money options our proposed model has a slight bias towards underpricing whereas LSM has bias towards overpricing.

Table 4 and 6 corresponding to prices obtained when initial volatility is set to long term average. Compared with the previous two tables, it is evident that long term average gives better results than setting the estimate as spot volatility for all the models. This can also be attributed to the fact that the estimated spot volatilities are greater than the long term average for almost all sample days which results in higher prices of the options. In this case our CV-Decomp approach performs better than the LSM approach for all moneyness-maturity groups except for deep-out-of-the-money options in which our proposed approach shows slight bias towards underpricing options.

The computation time for these two approaches shows a significant difference. The pricing calculation of about $8,177$ options took 24+ hours for LSM approach whereas for 3CV-Decomp took 38.812 seconds. Considering the pricing accuracy comparison between the two approaches, our constant–volatility decomposition approach is also efficient in this empirical study.

7 Conclusion

Empirical evidence supports the improvement provided by the stochastic volatility model of Heston (1993) in option pricing. Furthermore, this model yields a nearly closed-form formula for European options. However, parameter estimation for this model is very challenging as it involves terms associated with an unobservable process (volatility). Additionally, American option pricing in this model is significantly more difficult. While an expression relating an American option price to the corresponding European price is readily available, it also requires the determination of an early exercise premium that depends on the optimal early exercise surface. The latter is a function involving both time and instantaneous volatility, which can attain substantially different values, depending on the estimation technique used for the Heston specification. In this paper we develop a method that approximately determines the early exercise boundary, all the while resulting in a very efficient and accurate pricing procedure that is impervious to parameter estimation error in the Heston stochastic volatility model. Through an out-of-sample study, we also show that this procedure generates accurate market prices.

References

Adolfsson, T., Chiarella, C., Ziogas, A., and Ziveyi, J. 2013. Representation and numerical approximation of American option prices under Heston stochastic volatility dynamics. The University of Technology, Sydney.

Aït-Sahalia, Y. and Kimmel, R. 2007. Maximum likelihood estimation of stochastic volatility models. *Journal of Financial Economics*, 83:413–452.

AitSahlia, F., Goswami, M., and Guha, S. 2010a. American option pricing under stochastic volatility: An efficient and accurate numerical scheme. *Computational Management Science*, 7:171–187.

AitSahlia, F., Goswami, M., and Guha, S. 2010b. American option pricing under stochastic volatility: An empirical evaluation. *Computational Management Science*, 7:189–206.

AitSahlia, F. and Lai, T. 1999. A canonical optimal stopping problem for american options and its numerical solution. *Journal of Computational Finance*, 3(2):33–52.

AitSahlia, F. and Lai, T. 2001. Exercise boundaries and efficient approximations to american option prices and hedge parameters. *Journal of Computational Finance*, 4:85–103.

Bakshi, G., Cao, C., and Chen, Z. 1997. Empirical performance of alternative option pricing models. *Journal of Finance*, 52.

Bates, D. 2000. Post-87 crash fears in the s&p 500 futures option market. *Journal of Econometrics*, 94:181–238.

Bates, D. 2006. Maximum likelihood estimation of latent affine processes. *Review of Financial Studies*, 19:909–965.

Beliaeva, N. A. and Nawalkha, S. K. 2010. A simple approach to pricing American options under the Heston stochastic volatility model. *The Journal of Derivatives*, 17:25–43.

Black, F. and Scholes, M. 1973. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81:637–654.

Broadie, M., Detemple, J., Ghysels, E., and Torres, O. 2000. American options with stochastic dividends and volatility: A nonparametric investigation. *Journal of Econometrics*, 94:53–92.

Carr, P., Jarrow, R., and Myneni, R. 1992. Alternative characterizations of American put options. *Mathematical Finance*, 2:87–106.

CBOE 2015. *CBOE Market Statistics*
. Available at https://www.cboe.com/data/AnnualMarketStatistics.aspx.

Chernov, M. and Ghysels, E. 2000. A study towards a unified approach to the joint estimation of objective and risk neutral measures for the purpose of options valuations. *Journal of Financial Economics*, 56.

Chiarella, C., Ziogas, A., and Ziveyi, J. 2010. Representation of American option prices under Heston stochastic volatility dynamics using integral transforms. In *Contemporary Quantitative Finance* (C. Chiarella and A. Novikov, eds.).

Chockalingam, A. and Muthuraman, K. 2011. American options under stochastic volatility. *Operations Research*, 59:793–809.

Christoffersen, P., Jacobs, K., and Mimouni, K. 2010. Volatility dynamics for the S&P 500: Evidence from realized volatility, daily returns, and option prices. *Review of Financial Studies*, 23(8):3141–3189.

Clarke, N. and Parrott, K. 1996. The multigrid solution of two-factor american put options. *Tech. Rep. ,Oxford Computing Laboratory*.

Dragulescu, A. and Yakovenko, V. 2002. Probability distribution of returns under the Heston model with stochastic volatility. *Quantitative Finance*, 2:443–458.

El Karoui, N., Jeanbalc-Picqué, M., and Shreve, S. 1998. Robustness of the Black and Scholes formula. *Mathematical Finance*, 08:93–126.

Feller, W. 1951. Two singular diffusion problems. *Annals of Mathematics*, 54:173–182.

Fouque, J., Papanicolaou, G., and Sircar, R. 2000. *Derivatives in Financial Markets with Stochastic Volatility*. Cambridge University Press.

Gourieroux, C., Monfort, A., and Renault, E. 1993. Indirect inference. *J Appl Economet*, 8:s85–s118.

Heston, S. L. 1993. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies*, 6:327–343.

Hu, J. 2014. Does option trading convey stock price information. *Journal of Financial Economics*, 111:625–645.

Huang, J., Subrahmanyam, M. G., and Yu, G. G. 1996. Pricing and hedging american options: A recursive integration method. *Review of Financial Studies*, 3.

Huang, J.-Z.and Wu, L. 2004. Specification analysis of option pricing models based on time-changed levy processes. *Journal of Finance*, 7:1405–1439.

Ikonen, S. and Toivanen, J. 2007. Efficient numerical methods for pricing American options under stochastic volatility. *Numerical Methods for Partial Differential Equations*, 24:104–126.

Jacka, S. D. 1991. Optimal stopping and the american put. *Mathematical Finance*, 1:1–14.

Ju, N. 1998. Pricing of american option by approximating its early exercise boundary. *Review of Financial Studies*, 11.

Kim, I. J. 1990. The analytical approximation for the american options. *Review of Financial Studies*, 3:547–572.

Longstaff, F. A. and Schwartz, E. S. 2001. Valuing american options by simulation: A simple least-squares approach. *The Review of Financial Studies*, 14(1):113–147.

Medvedev, A. and Scaillet, O. 2010. Pricing American options under stochastic volatility and stochastic interest rates. *Journal of Financial Economics*, 98:145–159.

Muravyev, D., Pearson, N. D., and Brousssard, J. P. 2013. Is there price discovery in equity options. *Journal of Financial Economics*, 07:259–283.

Oosterlee, C. W. 2003. On multigrid for linear complementarity problems with application to american-style options. *Electron. Trans. Numer. Anal.*, 15:165–185.

Touzi, N. 1994. American options exercise boundary when the volatility changes randomly. *Appl. Math Optim*, 39:411–422.

Zvan, R., Forsyth, P. A., and Vetzal, K. R. 1998. A penalty method for american options with stochastic volatility. *Journal of Computational and Applied Mathematics*, 91:199–218.