Mean-Variance Spanning Tests
With Short-Sales Constraints

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Abstract

We address the implementation of Wald tests associated with mean-variance spanning when short positions are prohibited. We show how necessary and sufficient conditions for optimal mean-variance spanning differ from the sufficient conditions that have been implemented in testing 401(k) plans. We also exploit the characterization of stochastic discount factors in the presence of a risk-free asset to efficiently implement Wald tests in the context of mean-variance spanning with short-sales constraints.

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1. Introduction

A set of assets is said to span the mean-variance space if the efficient frontier they generate cannot be improved upon with additional assets (Huberman and Kandel, 1987). Mean-variance spanning has been used to assess the benefits of international diversification, evaluate mutual fund performance and test linear-factor asset pricing models (cf., Errunza et al., 1999; Li et al., 2003; De Roon and Nijman, 2001; Fama and French (2015, 2018); and the textbooks of Campbell et al., 1997 and Cochrane, 2005.) Elton et al. (2006) and Tang et al. (2010) use the concept of mean-variance spanning to evaluate defined contribution (DC) retirement plans, such as 401(k) plans in the United States. Investment funds in a DC plan are said to be mean-variance spanning if they generate the same efficient frontier as a benchmark set of index funds. In other words, their efficient frontier cannot be enhanced with one or more assets from the benchmark set. It is then likely that the plan will be able to satisfy a variety of risk/return preferences among its enrollees.

In their seminal paper, Huberman and Kandel (1987) allow for short-selling, but investment funds in DC plans cannot be held short. De Roon et al. (2001) introduce a Wald test for regression-based mean-variance spanning that accounts for short-sales constraints, which Elton et al. (2006) and Tang et al. (2010) apply to evaluate DC plans. However, we show that both sets of authors implement the test incorrectly. Specifically, they test a necessary, *but not sufficient*, condition for spanning. This leads to an overestimation of the number of plans that span as was empirically observed in AitSahlia et al. (2021).

For the implementation of their Wald test, De Roon et al. (2001) have to rely on mean estimates of unobservable stochastic discount factors. In this paper, we exploit the uniqueness of the mean stochastic discount factor in the presence of a risk-free asset, thereby avoiding potential issues related to covariance estimation and numerical instability.

The remainder of the paper is organized as follows. In Section 2, we re-examine the methodology of mean-variance spanning with short-sales constraints and re-derive the associated Wald tests using first principles of optimization. In Section 3, we show how the tests are incorrectly implemented in the extant literature on 401(k) plans, and in Section 4 develop new approaches for their proper implementation. Section 5 concludes.

2. A Re-Examination of Mean-Variance Spanning Tests

We review here the regression-based mean-variance spanning test methodology and highlight some issues related to its implementation. Along the way, we re-derive the main test in DeRoon et al. (2001) by avoiding their use of the stochastic discount factor approach for asset pricing and by relying instead on first principles for portfolio optimization. Due to the multitude of stochastic discount factors, DeRoon et al. (2001) suggest an implementation of their test that makes use of only the smallest and the largest mean discount factors. However, these are not observable and must be implied. As an alternative to their approach, we appeal to the uniqueness of the mean discount factor in the presence of a risk-free rate, with the former being the inverse of the latter. We also show that in their implementation of the Wald test provided by DeRoon et al. (2001), both Elton et al. (2006) and Tang et al. (2010) incorrectly specify their statistical hypotheses.

We start this review by first recalling some salient features from the classical set-up with *no short-sales constraints*, going back to at least Huberman and Kandel (1987). Suppose a 401(k) plan consists of a set of *K* funds. To assess the efficiency of this plan relative to a benchmark of *N* index funds, we want to determine whether the mean-variance frontier associated with the *K* funds coincides with that generated with the augmented set of *K*+*N* funds. In other words, the *K* funds are "sufficient" to span the frontier of the *K*+*N* funds.

Let R and r be the $K \times 1$ and $N \times 1$ return vectors of the *K* funds in the plan and the N benchmark indices, respectively. Denote by μ_R and μ_r their corresponding expected return vectors. Related covariance matrices are defined as follows:

$\Sigma_{\mathrm{R,R}}$: a $K \times K$ covariance matrix between the K funds captured by the returns vector *R*;

$\Sigma_{\mathrm{r,r}}$: an $N \times N$ covariance matrix between the returns of the *N* benchmark indexes captured by vector *r*;

$\Sigma_{\mathrm{R,r}}$: a $K \times N$ covariance matrix of the *K* fund returns with the *N* benchmark returns.

$\Sigma_{\mathrm{r,R}}$: an $N \times K$ covariance matrix of the N benchmark returns with the K funds. It is the transpose of $\Sigma_{R,r}$ (i.e., $\Sigma_{r,R} = \Sigma'_{R,r}$)[4]

These covariance matrices are then concatenated into the $(K + N) \times (K + N)$ covariance matrix Σ across all $K + N$ assets defined as:

$$\Sigma = \begin{pmatrix} \Sigma_{R,R} & \Sigma_{R,r} \\ \Sigma_{r,R} & \Sigma_{r,r} \end{pmatrix}.$$

Similarly, we denote by $\mu = \begin{pmatrix} \mu_R \\ \mu_r \end{pmatrix}$ the $(K + N) \times 1$ (concatenated) vector of expected returns across the $K + N$ assets.

A vector $\omega^* \in \mathbb{R}^{K+N}$ is mean-variance efficient across the *K*+*N* assets if it solves the optimization problem

$$\max_{\omega}\{\omega'\mu - \gamma\omega'\Sigma\omega - \eta(\omega' i - 1)\}, \tag{1}$$

where γ is a coefficient of risk aversion, i is the $(K + N)$-long unit vector and η is the Lagrange multiplier associated with the constraint $\omega' i = 1$.

The *K* funds in the plan are mean-variance spanning relative the *N* benchmark indexes if the optimal portfolio ω^* is allocated to only the *K* funds, and not to any of the *N* index assets, for all values of γ. That is, if ω^* is of the form:

$$\omega^* = \begin{pmatrix} \omega_R^* \\ 0_r \end{pmatrix},$$

where the *K*-vector ω_R^* is the allocation into the *K* funds and 0_r is an *N*-dimensional vector of zeros. The associated first-order conditions are then of the form:

$$\begin{pmatrix} \mu_R \\ \mu_r \end{pmatrix} - \eta \begin{pmatrix} i_R \\ i_r \end{pmatrix} = \gamma \begin{pmatrix} \Sigma_{R,R} & \Sigma_{R,r} \\ \Sigma_{r,R} & \Sigma_{r,r} \end{pmatrix} \begin{pmatrix} \omega_R^* \\ 0_r \end{pmatrix}, \tag{2}$$

[4] We use the superscript $'$ for matrix transposition.

for all values of η and γ, where i_R is the *K*-unit vector (same dimension as the vector *R*) and i_r is the *N*-unit vector (same dimension as the vector *r*). Solving for ω_R^* above yields:

$$\omega_R^* = \gamma^{-1}\Sigma_{R,R}^{-1}(\mu_R - \eta i_R).$$

Substituting this solution into the last *N* rows of the first-order conditions leads to

$$\mu_r - \eta i_r = \beta(\mu_R - \eta i_R),$$

where β is defined as $\Sigma_{r,R}\Sigma_{R,R}^{-1}$. Therefore, the *K* plans are mean-variance spanning relative the N benchmark indexes if, for all β and η, we have:

$$(\mu_r - \beta\mu_R) + (\beta i_R - i_r)\eta = \mathbf{0},$$

where $\mathbf{0}$ is a zero-vector of dimension $2N$.[5] In other words, the above is true if:

$$\begin{pmatrix} \mu_r - \beta\mu_R \\ \beta i_R - i_r \end{pmatrix} = 0, \tag{3}$$

If the funds in the plan span, then the optimal mean-variance allocation (ω_R^*, ω_r^*) is such that $\omega_r^* = \mathbf{0}_N$, where $\mathbf{0}_N$ is the N-dimensional zero vector. Incidentally, there is only one solution because the necessary assumption of positive definiteness of Σ leads to the convexity of the objective function (1).

If conditions (3) are satisfied, then any strategy ω_r involving the N-dimensional benchmark can be re-expressed as a strategy involving the K funds only. Indeed, for any ω_r involving the N indexes and satisfying $\omega_r' \cdot i_r = 1$, there exists a strategy ω_R involving the K funds only and such that $\omega_R' \cdot i_R = 1$, which is obtained as $\omega_r' \cdot \beta$ by virtue of the second set of equations in (3).

These theoretical conditions can be empirically tested, as originally proposed by Huberman and Kandel (1987) who test the null hypothesis

[5] For the remainder of the paper, we do not use boldface and omit the subscript to reference dimension if the context is evident.

$$\begin{pmatrix} \alpha \\ \beta i_R - i_r \end{pmatrix} = 0,$$

in the multivariate regression specification

$$r = \alpha + \beta R + \varepsilon, \tag{4}$$

where α is an $N \times 1$ vector and β is an $N \times K$ matrix.

2.1 Effect of Short-Sales Constraints

Given short-sales constraints, the mean-variance optimization problem across the *K*+*N* assets consists in determining the K-dimensional vector $\omega_R \geq \mathbf{0}_R$ and the *N*-dimensional vector $\omega_r \geq \mathbf{0}_r$ that maximize

$$(\omega_R', \omega_r') \begin{pmatrix} \mu_R \\ \mu_r \end{pmatrix} - \frac{1}{2}\gamma(\omega_R', \omega_r') \begin{pmatrix} \Sigma_{RR}\Sigma_{Rr} \\ \Sigma_{rR}\Sigma_{rr} \end{pmatrix} \begin{pmatrix} \omega_R \\ \omega_r \end{pmatrix}, \tag{5}$$

subject to $\omega_R' \cdot i_R + \omega_r' \cdot i_r = 1$.

The corresponding Karush-Kuhn-Tucker (KKT) are then:

$$\begin{pmatrix} \mu_R \\ \mu_r \end{pmatrix} - \gamma \begin{pmatrix} \Sigma_{RR}\Sigma_{Rr} \\ \Sigma_{rR}\Sigma_{rr} \end{pmatrix} \begin{pmatrix} \omega_R \\ \omega_r \end{pmatrix} - \eta \begin{pmatrix} i_R \\ i_r \end{pmatrix} + \begin{pmatrix} \delta_R \\ \delta_r \end{pmatrix} = 0,$$

where the Lagrange multiplier η is a scalar associated with the constraint $\omega_R' i_R + \omega_r' i_r = 1$, and δ_R and δ_r are non-negative Lagrange multipliers of dimensions K and N, respectively, that are associated with the no-short sales constraints $\omega_R \geq 0$ and $\omega_r \geq 0$. The latter multipliers satisfy the complementarity slackness conditions:

$$\delta_{R,j}\omega_{R,j} = 0, \text{ for } 1 \leq j \leq K, \quad \text{and} \quad \delta_{r,j}\omega_{r,j} = 0, \quad \text{for } 1 \leq j \leq N.$$

If the funds in the plan span, then the optimal mean-variance allocation (ω_R^*, ω_r^*) is such that $\omega_r^* = 0_N$. As a result, the KKT conditions become:

$$\mu_R - \gamma \Sigma_{RR}\omega_R^* - \eta i_R + \delta_R = 0 \tag{6}$$

$$\mu_r - \gamma\Sigma_{rR}\omega_R^* - \eta i_r + \delta_r = 0 \quad (7)$$

$$\delta_{R,j}\omega_{R,j}^* = 0, \text{ for } 1 \leq j \leq K;$$

$$\omega_R^* \geq 0_R, \quad \delta_R \geq 0_R$$

$$\omega_R^{*\prime} \cdot i_R = 1.$$

With one eye toward the specification (4) for the empirical tests that will follow, let $\alpha = \mu_r - \beta\mu_R$. Then we can rewrite (7) as

$$\alpha + \beta\mu_R - \gamma\Sigma_{rR}\omega_R^* - \eta i_r + \delta_r = 0. \quad (8)$$

From (6), we have, given that Σ_{RR} is positive definite,

$$\omega_R^* = \frac{1}{\gamma}\Sigma_{RR}^{-1}(\mu_R - \eta i_R + \delta_R). \quad (9)$$

Note that $\delta_{R,j} = 0$ for $\omega_{R,j}^* > 0$. For notational simplicity, and to avoid introducing an additional index as in De Roon et al. (2001), we may assume that $\omega_R^* > 0$; i.e., for all subscripts. As a result, the optimal allocation $\omega_{R,j}^* > 0$ can now be written as:

$$\omega_R^* = \frac{1}{\gamma}\Sigma_{RR}^{-1}(\mu_R - \eta i_R). \quad (10)$$

Substituting (10) back in (8), we get:

$$\alpha + \beta\mu_R - \Sigma_{rR}\Sigma_{RR}^{-1}(\mu_R - \eta i_R) - \eta i_r + \delta_r = 0. \quad (11)$$

Again, with an eye toward the specification (4) for the forthcoming empirical tests, let $\beta = \Sigma_{rR}\Sigma_{RR}^{-1}$. Then (11) can now be re-written as

$$\alpha + \beta\mu_R - \beta(\mu_R - \eta i_R) - \eta i_r + \delta_r = 0,$$

which becomes

$$\alpha + \eta(\beta i_R - i_r) + \delta_r = 0. \quad (12)$$

Since $\delta_R \geq 0$, (12) implies the necessary condition

$$\alpha + \eta(\beta i_R - i_r) \leq 0. \quad (13)$$

Condition (13) is also sufficient for (12) since $\delta_r \geq 0$ is uniquely characterized by (12). In other words, if (13) is true, then setting

$$\delta_r = -[\alpha + \eta(\beta i_R - i_r)]$$

will make (12) true (a tautology). Note that (13) is the same as (15) on p. 727 in De Roon et al. (2001), once we divide through our expression by η, which is the inverse of their υ.[6] Finally, multiplying both sides of (9) by $\omega_R^{*\prime}$ and using $\omega_R^{*\prime} \cdot i_R = 1$ and $\omega_R^{*\prime} \cdot \delta_R = 0$ yields

$$\omega_R^{*\prime} \cdot \mu_R - \gamma \omega_R^{*\prime} \Sigma_{RR} \omega_R^* - \eta = 0.$$

With the expression (9) above for ω_R^*, the latter equation becomes

$$\frac{1}{\gamma}(\mu_R - \eta i_R)' \Sigma_{RR}^{-1} \eta i_R - \eta = 0.$$

Assuming $\gamma < \mu_R' \Sigma_{RR}^{-1} i_R$, the Lagrange multiplier η for the portfolio allocation that satisfies:

$$\eta = \frac{\mu_R' \Sigma_{RR}^{-1} i_R - \gamma}{i_K' \Sigma_{RR}^{-1} i_R}.$$

The assumption above is equivalent to $\eta > 0$, which is implicit in DeRoon et al. (2001). For ease of comparison with De Roon et al. (2001), we multiply through (13) by $\upsilon = 1/\eta > 0$ to rewrite it as

$$\upsilon\, \alpha + \beta i_R - i_r \leq 0. \quad (14)$$

We should note that the assumption of optimal weights in the funds being strictly positive ($\omega_R^* > 0$) can be relaxed. For a given risk tolerance parameter γ, denote by $\mathcal{B}$ the set of indices out of the K funds in the plan so that for the spanning solution $(\omega_R^*, 0_N)$ satisfying the the KKT conditions (6)-(7), we have $\omega_{R,j}^* = 0$ for i ∈ $\mathcal{B}$.

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[6] We again recall here that we do not use the equivalent of their superscript (υ) for notational simplicity, but we similarly refer here to the dimensions of α and, later, to $\mathcal{B}$, associated with assets with non-zero, i.e. positive allocation.

Also, let $\overline{\mathcal{B}}$ be the complement of $\mathcal{B}$ in $\{1,2,\dots,K\}$. In this case, $\delta_{R,j} = 0$, for $j \in \overline{\mathcal{B}}$. Then, from (9), we can easily show, similarly to (13), that spanning is equivalent to

$$\alpha + \check{\beta}(\mu_R - \eta i_R) + \eta(\beta i_R - i_r) \leq 0,$$

where $\check{\beta} \equiv \Sigma_{r\mathcal{B}.\mathcal{B}} \times \Sigma_{\mathcal{B},RR}^{-1}$, with $\Sigma_{r\mathcal{B}.\mathcal{B}}$ and $\Sigma_{\mathcal{B},RR}^{-1}$ consisting of the columns in $\mathcal{B}$ and rows in $\mathcal{B}$ of, respectively, the matrices Σ_{rR} and Σ_{RR}^{-1}. The difference between the above expression and (13) is the presence of $\check{\beta}(\mu_R - \eta i_R)$, where it vanishes in (13). One can make the case that $\check{\beta}(\mu_R - \eta i_R)$ is negligible as the inequality above has to apply to all risk factors. As a result, the binding indices would have to be the same for all risk parameters, which for a large number of funds in a plan, would likely mean that $\mathcal{B}$ is empty.

3 A Critique of the Current Literature on 401(k) Plan Spanning Tests

In the presence of short-sales constraints, both Elton et al. (2006) and Tang et al. (2010) use the *implicit* optimality condition

$$\mu_r - \beta\mu_R \leq \mathbf{0}, \tag{15}$$

where $\mathbf{0}$ is, this time, of dimension N. While not using any formal optimization model[7], Elton et al. (2006) justify their choice by arguing that "*If short sales are forbidden, then only the addition of an asset with positive alpha can improve the efficient frontier…*" (p. 1304). Similarly, Tang et al. (2010) state: "*As short-sales are not allowed for market benchmark index, if none of the α_i are statistically significantly positive, we could conclude that performance of funds under the plan cannot be improved by holding a long position in any of the eight market benchmark indices*." (p. 1078).

[7] We infer their implicit optimality condition (15) on the basis of their exclusive reliance on the specification (4) and their null hypothesis $\alpha \leq 0$.

We argue next that (15) is in fact not reflective of an optimality condition with short-sales constraints, in the sense that one can still improve the efficient frontier of a given set with the addition of another asset whose alpha is negative relative to the former. Consider assets 1, 2, and 3 with return covariance matrix

$$\begin{pmatrix} 0.011 & 0.002 & 0.001 \\ 0.002 & 0.012 & 0.003 \\ 0.001 & 0.003 & 0.020 \end{pmatrix}$$

and their associated expected returns 0.043, 0.001, and 0.028, respectively. Then the mean return of asset 2 (call it μ_2) satisfies $\mu_2 = 0.001 = -0.012 + 0.1689 \times \mu_1 + 0.1416 \times \mu_3$, where $\mu_1 = 0.043$ and $\mu_3 = 0.028$ are the mean returns of assets 1 and 3, respectively. For a given expected return of 3%, the variance minimizing allocation when only assets 1 and 3 are considered is 0.1333 and 0.8667, with a resulting standard deviation of portfolio return equal to 0.1243. On the other hand, with the addition of asset 2, and for the same level of expected return of 3%, the variance minimizing strategy is 0.5410, 0.2265, and 0.2326, for assets 1, 2, and 3, respectively. The standard deviation of the return on this portfolio is smaller, namely, 0.0773, despite the negative alpha of the additional asset 2.

This example clearly shows that the condition (15) advocated by Elton et al. (2006) and Tang et al. (2010) is not sufficient for spanning. Observe that this example is compatible with a risk-aversion coefficient $\gamma = 19.6$, which results in a zero-beta rate $\eta = -8.7\%$. These parameters and the associated strategy above do not satisfy the necessary and sufficient optimality conditions under short sales constraints (6)-(7), specifically,

$$\mu_r - \beta(\mu_R - \eta i_R + \delta_R) - \eta i_r \leq 0, \tag{16}$$

the left-hand of which in our counter-example translates to

$$.001 - (0.1689, 0.1416) \times ((0.043, 0.028)' + 0.0873 \times i_2 + 0_2) + 0.0873 = 0.05 > 0,$$

where $i_2 = (1,1)'$ and $0_2 = (0,0)'$, which shows that Asset 2 does indeed enter in the three-asset portfolio with expected return of 3% (no spanning). Clearly, condition (16) subsumes (15), and the latter is only necessary, but not sufficient. As argued earlier, conditions (6)--(7) are ultimately captured equivalently by conditions (13).

4 Wald Test Implementation Challenges

For spanning, conditions (13), or their De Roon et al. (2001) version (14), have to be satisfied for the entire range of values of the mean discount factor υ. Within the framework of the regression model (4), De Roon et al. (2001) suggest (pp. 729-730) that it is enough to jointly test

$$\begin{aligned} 1\,\alpha + \beta i_R - i_r &\leq 0 \\ \upsilon_{min}\,\alpha + \beta i_R - i_r &\leq 0, \end{aligned} \tag{18}$$

where 1 is the upper bound on the values of υ and their lower bound is $\upsilon_{min} = \frac{1}{E[R^{GMV}]}$, where $E[R^{GMV}]$ is the mean return of the global minimum variance portfolio.

The inequalities above restrict linear transformations on the N elements of α and the $N \times K$ elements of β. Following Kodde and Palm (1986), as suggested by De Roon et al. (2001), a Wald statistic can be used to test the inequalities in (18), namely

$$\xi = \min_{\gamma \geq 0} (\tilde{\gamma} - \gamma)' \tilde{\Sigma}^{-1} (\tilde{\gamma} - \gamma), \tag{19}$$

where

$$\tilde{\gamma} = \begin{pmatrix} -\hat{\alpha} - \hat{\beta} \times i_R + i_r \\ -\frac{1}{1+\mu}\hat{\alpha} - \hat{\beta} \times i_R + i_r \end{pmatrix}, \tag{20}$$

with $\mu = E[R^{GMV}] - 1$, and

$$\tilde{\Sigma} = \begin{pmatrix} -I_N & -A \\ -\dfrac{1}{1+\mu} I_N & -A \end{pmatrix} \Omega \begin{pmatrix} -I_N & -A \\ -\dfrac{1}{1+\mu} I_N & -A \end{pmatrix}', \tag{21}$$

with I_N defined as the $N \times N$ identity matrix, A as the Kronecker product $I_N \otimes i_R{}'$, and Ω as the $(N + NK) \times (N + NK)$ covariance matrix between the multivariate intercept α and the loading matrix β in the multivariate regression (4). Using standard notation, $\hat{\alpha}$ and $\hat{\beta}$ refer to estimates of α and β, respectively.[8] We observe that virtually all, if not all, plans offer risk-free funds and that all assets are priced in a context where a risk-free asset is present for asset pricing consideration. As a result, we instead appeal to the fact that in the presence of a risk-free rate, say r_f, there is only one mean stochastic discount factor, namely $\frac{1}{1+r_f}$. Consequently, instead of the two sets of inequalities in (18), we only need to deal with one set in (17) with $\upsilon = \frac{1}{1+r_f}$, and for (20)—(21), we now have:

$$\tilde{\gamma} = -\frac{1}{1+r_f}\hat{\alpha} - \hat{\beta} \times i_R + i_r$$

$$\tilde{\Sigma} = \left(\frac{1}{1+r_f} I_N \quad - A\right) \Omega \left(\frac{1}{1+r_f} I_N \quad - A\right)' \tag{22}$$

It should be noted that an additional benefit of the reliance on the characterization of stochastic discount factors in the presence of a risk-free asset is that it avoids the use, as advocated by DeRoon et al. (2001), of estimating the covariance matrix between a very large number of assets in order to arrive at a value for υ_{min}. As pointed out by DeMiguel et al. (2009), this estimation exercise is not trivial at all and is in fact at the source of the difficulty of consistently dominating the simple 1/n strategy.

5. Conclusion

Regression-based mean-variance spanning tests are ubiquitous in empirical finance. Our paper centers on challenges that arise when these tests are implemented in the context of short-sales constraints, as in defined-contribution retirement plans such as the 401(k) plans for U.S. employees. While the standard Wald testing methodology relies on the implied means of unobservable discount factors, we exploit the property that the mean

[8] As a reminder, α and β are vectors of dimension $N \times 1$ and $NK \times 1$, respectively.

discount factor is uniquely determined in the presence of a risk-free asset to help with its efficient implementation.

References

AitSahlia, F, Doellman, T., Sardali, S., 2021. Menu simplification for 401(k) plans, working paper, Warrington College of Business, University of Florida.

Campbell, J. Y, Low, A., McKinlay, A., 1997. The econometrics of financial markets. Princeton University Press, Princeton, New Jersey.

Cochrane, J. H., 2005. Asset pricing. Princeton University Press, Princeton, New Jersey.

De Miguel, V., Garlappi, L., Uppal, R., 2009. Optimal versus naive diversification: How inefficient is the 1/n portfolio strategy? *Review of Financial Studies* 22(5), 1915-1953.

De Roon, F., Nijman, T., Werker, B., 2001, Testing for the mean-variance spanning with short sales constraints and transaction costs: The case of emerging markets. *Journal of Finance* 56(2), 721-742.

De Roon, F.A., Nijman, T. E., 2001. Testing for mean-variance spanning: A survey. *Journal of Empirical Finance* 8, 111-155.

Elton, E., Gruber, M., Blake, C., 2006. The adequacy of investment choices offered by 401(k) plans. *Journal of Public Economics* 90, 1299–1314.

Errunza, V., Hogan, K., Hung, M., 1999. Can the gains from international diversification be achieved without trading abroad? *Journal of Finance* 54, 2075–2107.

Fama, E., French, K., 2015. A five-factor asset pricing model. J. Financ. Econ., 116, pp. 1-22.

Fama, E., French, K., 2018. Choosing factors. J. Financ. Econ., 128 (2), pp. 234-252.

Huberman, G., Kandel, S., 1987. Mean-variance spanning. *Journal of Finance* 42, 873– 888.

Kodde, D., Palm, F., 1986. Wald criteria for jointly testing equality and inequality restrictions. *Econometrica* 56, 1243–1248.

Tang, N., Mitchell, O., Mottola, G., Utkus, S., 2010. The efficiency of sponsor and participant portfolio choices in 401(k) plans. *Journal of Public Economics* 94, 1073-1085.

Wolak, F., 1989. Testing inequality constraints in linear econometric models. *Journal of Econometrics* 41, 205–235