Discretely Monitored Look-Back Option Prices and their Sensitivities in Lévy Models

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Abstract

We present an efficient method to price discretely monitored lookback options when the underlying asset price follows an exponential Lévy process. Our approach extends the random walk duality results of AitSahlia and Lai (1998) originally developed in the Black-Scholes set-up and exploits the very fast numerical scheme recently developed by Linetsky and Feng (2008, 2009) to compute and invert Hilbert transforms. Though Linetsky and Feng (2009) do apply these transforms to price lookback options, they require an explicit transition probability density of the Lévy process and impose a condition that excludes the pure jump variance gamma process, among others. In contrast, our approach is much simpler and makes use of only the characteristic function of the log-increment, which is central to Lévy processes. Furthermore, by focusing our approach on determining the distribution function of the maximum of the Lévy process we can also determine price sensitivities with minimal additional computational effort.

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1 Introduction

Lookback options provide the largest payoff potential because their holders can choose (in hindsight) the exercise date with the advantage of having full path information. Lookback options were initially devised mainly for speculative purposes but starting with currency markets, their adoption has been increasing significantly, especially in insurance and structured products during the past decade. For example, embedded lookback option features have been highlighted in equity-indexed annuities (cf. Tiong(2000), Gerber and Shiu (2003), and Lee (2003).)

Lookback option payoffs depend on the maximum or minimum price of the underlying asset observed over the contract period. The holder of a floating–strike call gets the right to buy at the lowest price attained over the contract period and sell at the price on the expiration date. The holder of a floating–strike put gets the right to buy at the price on the maturity date and sell at the highest price over the contract period. For fixed–strike lookback options, the holder of a call option gets the right to buy the security at a fixed price and sell at the highest price over the contract period. The holder of a fixed–strike lookback put gets the right to buy at a fixed price and sell at the lowest price attained by the underlying over the contract period. Other variations on these basic forms of lookback options include versions for which the maximum or minimum is replaced by a fraction of it and others where the monitoring periods are smaller intervals of the contract duration.

When the underlying asset price is continuously monitored, analytical and closed-form option pricing expressions have been advanced, starting with the founding papers of (Conze & Viswanathan, 1991) and (Goldman et al., 1979) for the standard Black-Scholes model. Extensions include the CEV model of Davydov and Linetsky (2001), who made use of Laplace transforms, and Merton's jump-diffusion model, with normally distributed jumps occurring according to a Poisson process, as developed in Broadie and Yamamoto (2003) who further extend their fast–Gauss transform approach to jumps with double-exponential distributions (Broadie & Yamamoto, 2005).

In practice, monitoring occurs at predetermined discrete dates (once every day or every week, for example) and the ensuing pricing is mathematically and computationally challenging. Substantial mis-pricing occurs when a discretely monitored contract is priced approximately by a continuous-monitoring formula (cf. (Broadie et al., 1999) ,(Heynen & Kat, 1995).) (Broadie et al., 1999) introduce correction terms so that the continuous-monitoring formulas can be used as approximations for the discretely monitored options. (AitSahlia & Lai, 1998) use the duality property of random walks to derive recursively the distribution of the extrema of the geometric Brownian motion price process and determine the price of the lookback option by numerical integration. (Andricopoulos et al., 2003) propose a procedure exploiting properties of Brownian motion to improve methods based on trees and finite-differences. Beyond the Black-Scholes model, (Petrella & Kou, 2004) find Laplace transforms of discrete lookback options using a recursion based on Spitzer's formula. They invert the Laplace transforms numerically to get the lookback option price and hedging parameters for several Lévy price models. More recently, (Feng & Linetsky, 2009) developed a forward recursion on the prices of the lookback option utilizing Hilbert transforms and Fourier transforms. In particular, they propose a very efficient algorithm to invert such transform with a complexity comparable to the widely popular Fast Fourier Transform. However, their approach is restricted by some conditions making it inapplicable to important pure-jump processes, such as the popular variance gamma (Madan and Seneta (1990).). Even more recently, Green et al. (2010) exploit the Weiner–Hopf technique to derive analytic expressions reminiscent of the classical Spitzer formulas. Their approach requires that distributions have explicit density functions and that their characteristic functions be analytic. In contrast, our method works directly on the characteristic functions of the asset price change, with minimal assumptions, making it more generally applicable to pure jump processes, which are excluded from the previous two papers. In addition, our approach has the same efficient computational complexity of Feng and Linetsky (2009) as it also uses their fast algorithm for the evaluation and inversion of Hilbert and Fourier transforms. Furthermore, our approach is such that price sensitivities are readily obtainable with little additional computational cost. For the remainder, our paper is organized as follows. Section 2 highlights the prominence of jumps in underlying asset prices and reviews their most general and practical classification as Lévy models. Section 3 presents an overview of the fast inversion algorithm developed by Feng and Linetsky for Hilbert transforms. Section 4 presents our extension of earlier random walk duality results to Lévy processes, which are then exploited in Sections 5 and 6 to price fixed– and floating–strike lookback options, respectively. Further extensions are presented in Section 7 and Section 8 concludes.

2 Jumps in asset prices and Lévy models

In the classical Black-Scholes model, the underlying asset price is assumed to follow a diffusion process of the form

$$dS_t = S_t \mu dt + S_t \sigma dW$$

where S_t is its price time t, μ its mean return rate, σ its volatilty and W a Wiener process. The main strength of this model is its resulting option pricing formula (Black and Scholes (1973).) However, return distributions have long been known to exhibit leptokurtic and asymmetric features that differ markedly from the normal distribution (cf., for example, Mandelbrot (1963) and Fama (1965), for some of the classical evidence, and Cont and Tankov (2004) and Wu (2008) for a more recent perspective.)

One of the earliest models capturing fatter tails than the normal distribution for asset returns is the jump-diffusion model of (Merton, 1976), where the asset price follows the stochastic differential equation

$$dS_t = S_t \mu dt + S_t \sigma dW + dq,$$

where q is a Poisson process with normally distributed jumps that are independent of W. These random jumps lead to asset price return distributions with fatter tails than the normal distribution. Subsequent models capturing additional features such as asymmetry include the popular double-exponential jump–diffusion of (Kou, 2002), where jumps occur as in Merton (1976) but follow an asymmetric double Laplace (exponential) distribution, and the pure–jump models of Madan and Seneta (1990) and (Carr et al., 2002). These models are now recognized as part of the much larger and more general class of exponential Lévy models.

A process $(X_t)_{t>0}$ is called a Lévy process if it has

(a) Independent increments, that is for all $t_0, t_1, ..., t_n$, the random variables $X_{t_0}, X_{t_1} - X_{t_0}, ..., X_{t_n} - X_{t_{n-1}}$ are independent

(b) Stationary increments, that is $X_t - X_s$ has the same distribution as $X_{t-s+u} - X_u$ and

(c) Continuous paths a.e., that is $\lim_{h \to 0} P(|X_{t+h} - X_t| \geq \epsilon) = 0$.

Every Lévy process can be fully described by three parameters. The first two, a and σ^2, characterize the continuous component of the Lévy process, and the third, $\nu(x)$, is a function called the Lévy density that is associated with the jump component of the Lévy process. Furthermore, a is the constant drift rate of the continuous component and σ^2 is the constant variance of the continuous component.

Using only those three parameters, the Lévy-Khinchine representation formula (cf. Sato (1999)):

$$ln\mathbb{E}[e^{i\theta X_t}] = ait\theta - \frac{1}{2}\sigma^2 t\theta^2 + t\int (e^{i\theta x} - 1 - i\theta x\mathbf{I}_{|x|<1})\nu(x)dx$$

where $a\epsilon\mathbb{R}, \sigma \geq 0$ and $\int_{\mathbb{R}/0} min\{1, x^2\}\nu(x)dx < \infty$, allows for an easy retrieval of the characteristic function, $\phi(\theta) = \mathbb{E}[e^{i\theta X_t}]$, which makes them practically appealing, given the readily available Fourier transform inversion technology (cf. . Carr & Madan(1999) and Lee (2004).)

Lévy processes can have either finite activity, with a finite number of jumps within a given time interval, or infinite activity, with an infinite number of jumps and no diffusion within any interval. However, pure jump processes ($\sigma = 0$) with infinite activity, can often be difficult to distinguish from pure diffusion processes (Wu (2008).) When $\nu(x)$ is zero we have a pure diffusion process. The arrival rate for jumps is given by

$$\int_{\mathbb{R}/0} \nu(x)dx = \lambda$$

If $\lambda < \infty$, then the mean arrival rate of jumps is finite, and when $\lambda = \infty$ the number of jumps over any interval will be infinite. The simplest Lévy process is the Brownian motion, for which $\nu(x) = 0$. The processes discussed earlier can be identified as Lévy processes with explicit expressions for the triplets appearing in the Lévy-Khinchinee representation (see, e.g., Cont and Tankov (2004).)

Exponential Lévy models entail the representation of the underlying asset price S_t of an option as $S_t = S_0 \exp\{X_t\}$, where S_0 is its initial price and X_t is a Lévy process such that $X_0 = 0$. The value of a (*vanilla*) option with payoff function $h(S_T)$ on expiration date T is $\mathbb{E}[h(S_T)]$ and can be calculated very efficiently via fast Fourier transform techniques as described in (Carr & Madan, 1999).

3 The fast Hilbert transform technique

The valuation of discretely monitored, path-dependent options invariable require some form of recursion and exploitation of random walk results. It is thus not surprising to find, among the first efforts to price discrete lookback options, the involvement Spitzer's classic identity, as was done in Petrella and Kou (2004), for the double-exponential jump–diffusion model of Kou (2002), and Borovkov and Novikov (2002) for more general Lévy models. However, the computational complexity of this approach, which is quadratic in the number of monitoring dates, can become prohibitive for the typical practice of daily monitoring. A better alternative is the method of Feng and Linetsky (2009), based on Hilbert transforms and the so-called sinc expansion, which grows linearly in terms of number of monitoring dates and comes with an approximation error for the Hilbert transform that decays exponentially fast in its grid size. Their algorithm is in effect the Hilbert transform version of the classical fast Fourier transform, a close relative of the former. However, for their approach to work, they need to impose a condition (see their expression (2.2) on page 505) that excludes certain pure jump processes such as the popular variance (VG) of Madan and Seneta (1990). Our approach, on the other hand, makes use of random walk duality and preserves the computational complexity of Feng and Linetsky (2009) without imposing their restriction. Before we describe it fully in the next sections, we begin with a brief review of the Hilbert transform and the fast inversion algorithm of Feng and Linetsky (2009). The need to resort to the Hilbert transform instead of working with the Fourier transform in the context of lookback (or related barrier) options stems from the requirement of having to compute the Fourier transform of products of the form $\mathbf{1}_{(0,\infty)} \times f$, where $\mathbf{1}_{(0,\infty)}$ is the indicator function for the interval $(0, \infty)$ and f is a function satisfying some integrability condition. Specifically, we have (see Stenger (1993), for example):

$$\mathcal{F}\left(\mathbf{1}_{(0,\infty)} \cdot \phi\right)(\xi) = \frac{1}{2}\hat{\phi} + \frac{i}{2}\mathcal{H}\left(\hat{\phi}\right)(\xi),$$

for a function $\phi \in \mathbf{L}^p(\mathbb{R})$, $1 < p < \infty$ (or $\phi \in \mathbf{L}^1(\mathbb{R})$ such that $\hat{f} \in \mathbf{L}^1(\mathbb{R})$, where $\hat{f}$ is the Fourier transform of f defined as

$$\hat{f}(\xi) \equiv \mathcal{F}(f)(\xi) = \int_{\mathbb{R}} e^{i\xi x} f(x)dx \qquad \text{for } \xi \in \mathbb{R}, \tag{1}$$

and $\mathcal{H}(f)$ is the Hilbert transform of $f \in \mathbf{L}^p(\mathbb{R})$, $1 \leq p < \infty$, defined a.e. as the Cauchy principal value integral

$$\mathcal{H}(f)(x) = \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{f(y)}{x - y} dy \tag{2}$$

The classical fast Fourier transform (FFT) algorithm enables us to discretize and invert a Fourier transform with a computational complexity of $O(N \log_2 N)$, where N is the number of points in the discretization. This is an advantage over computing the DFT in the naive way, which results in a complexity of $O(N^2)$. Feng and Linetsky (2008, 2009) use Whittaker cardinal series (Sinc expansion) to approximate $\mathcal{H}$ with

$$\mathcal{H}(f)(\xi) \approx H_{h,M} f(\xi) = \sum_{m=-M}^{M} f(mh) \frac{1 - cos[\pi(\xi - mh)/h]}{\pi(\xi - mh)/h},$$

where h is the discretization step size and $M > 0$ is the truncating integer for the integral approximation. After this discretization step, they then use the FFT and Toeplitz matrix-vector multiplication to compute $H_{h,M} f(\xi)$. The overall computational complexity to find the Hilbert transform is $O(M \log_2 M)$, or the same complexity as the FFT for the Fourier transform. Furthermore, the error in the approximation decays exponentially as h is taken smaller. The resulting discretization–inversion algorithm for Hilbert transforms therefore has a computational complexity of $O(NM \log_2 M)$ when there are N discrete monitoring dates. The application of this computational tool in the approach of Feng and Linetsky (2009) requires that the transition probability density of the Lévy process be known explicitly and excludes important price specifications such as the variance gamma process. In our approach, we still manage to take advantage of the computational complexity of their Hilbert transform inversion algorithm while avoiding their restrictive constraints.

4 Duality and Extrema of Random Walks

Under the assumption that the underlying price $\{S_t\}$ follows an exponential Lévy process and given the discrete monitoring of the maximum and minimum at dates $t_1, t_2, \ldots, t_N$, we can write $S_{t_n} = S_0 e^{U_n}$, where $\{U_n : n \geq 1, U_0 = 0\}$ is a random walk with i.i.d. increments X_i such that their common characteristic function $\hat{\Psi}$ is explicitly known thanks to the Lévy-Khinchinee formula.

Given N discrete monitoring dates $t_1, t_2, \ldots, t_N$, the maximum price $\tilde{M}_N = \max\{S_{t_1}, \ldots, S_{t_N}\}$ and minimum price $\tilde{\Lambda}_N = \min\{S_{t_1}, \ldots, S_{t_N}\}$ of the underlying asset lead to inception (time $t_0 = 0$) prices for both fixed strike and floating strike lookback options summarized in the Table 1 below.

Table 1: Loookback option prices at time $t_0 = 0$

Fixed strike	Floating strike
$e^{-rT} E\left(\tilde{M}_N - K\right)^+$	$e^{-rT} E\left(\tilde{M}_N - S_{t_N}\right)^+$
$e^{-rT} E\left(K - \tilde{\Lambda}_N\right)^+$	$e^{-rT} E\left(S_{t_N} - \tilde{\Lambda}_N\right)^+$

The difficulty in pricing these options is essentially due to the fact that the distributions of $\tilde{M}_N$ and $\tilde{\Lambda}_N$ are not known in analytical form even for the standard geometric Brownian motion of the Black-Scholes model.

Define now $\tau_- = \inf\{n : U_n \leq 0\}$ to be the first passage of the log-price process below zero, observed on a monitoring date, and $\tau_+ = \inf\{n : U_n > 0\}$ the corresponding first passage of the log-price process above zero. τ_- or τ_+ are called 'ladder epochs'. The duality property of this random walk will enable us, through τ_- and τ_+, to derive recursive expressions leading to the distributions of the extrema $\tilde{M}_N$ and $\tilde{\Lambda}_N$.

From (AitSahlia & Lai, 1998) we know that the distribution of the maximum log-price can be written as

$$\begin{aligned}
&P\{M_N \in dx\} = \\
&\quad P\{U_1 \in dx\}\, P\{X_2 \le 0, X_2 + X_3 \le 0, \ldots, X_2 + \cdots X_N \le 0\} \\
&\quad + \sum_{\nu=2}^{N} \Big[P\{U_\nu > U_i, i < \nu; U_\nu \in dx\} \times \\
&\qquad\quad P\{X_{\nu+1} \le 0, X_{\nu+1} + X_{\nu+2} \le 0, \ldots, X_{\nu+1} + \cdots + X_N \le 0\} \Big]
\end{aligned}$$

for $x > 0$. Furthermore, the duality of random walks (Feller, 1971), lets us rewrite one of the above probabilities in terms of one of the ladder epochs

$$\begin{aligned}
&P\{U_\nu > U_i, i < \nu; U_\nu \in dx\} \\
&\quad = P\{U_\nu - U_{\nu-1} > 0, \ldots, U_\nu - U_1 > 0; U_\nu \in dx\} \\
&\quad = P\{U_1 > 0, \ldots, U_{\nu-1} > 0; U_\nu \in dx\} \\
&\quad = P\{\tau_- > \nu; U_\nu \in dx\}
\end{aligned}$$

And another of the above probabilities can also be written in terms of one of the ladder epochs

$$\begin{aligned}
&P\{X_{\nu+1} \le 0, X_{\nu+1} + X_{\nu+2} \le 0, \ldots, X_{\nu+1} + \cdots + X_N \le 0\} \\
&\quad = P\{U_1 \le 0, U_2 \le 0, \ldots, U_{N-\nu} \le 0\} \\
&\quad = P\{\tau_+ > N - \nu\}
\end{aligned}$$

Putting the simplified probabilities into the original equation yields, for $x > 0$,

$$\begin{aligned}
P\{M_N \in dx\} &= P\{U_1 \in dx\}\, P\{\tau_+ > N - 1\} \\
&\quad + \sum_{\nu=2}^{N} P\{\tau_- > \nu; U_\nu \in dx\}\, P\{\tau_+ > N - \nu\}
\end{aligned} \tag{3}$$

And for $x = 0$, it is clear that $P\{M_N = 0\} = P\{\tau_+ > N\}$. The advantage of writing the above probabilities in terms of the ladder epochs τ_- and τ_+ is that they can be determined recursively.

In contrast to the standard definitions (1) and (2) above for the Fourier and Hilbert transforms, resp., used by Feng and Linetsky (2009) we opt for slightly more general forms. Specifically, define now the Fourier transform or characteristic function of a distribution function F of a real random variable X as (cf. Chung (1974)) as:

$$\mathcal{F}(F)(\xi) = E\left(e^{i\xi X}\right) = \int_{\mathbb{R}} e^{i\xi x} dF(x).$$

Alternatively, the notation $\widehat{F}$ will also be used. Furthermore, we define the Hilbert transform for such F by the Cauchy principal value integral

$$\mathcal{H}(F)(\xi) = \frac{1}{\pi} P.V. \int_{\mathbb{R}} \frac{dF(x)}{\xi - x},$$

which reduces to the earlier definition of a Hilbert transform when F is absolutely continuous (with respect to the Lebesgue measure) with a density $f \in L^p(\mathbb{R})$. We can now state the following generalization to Proposition 1 in AitSahlia and Lai (1998).

Proposition 1. Let J be either $(0, \infty)$ or $(-\infty, 0]$ and $\tau = \inf\{n : U_n \notin J\}$. For $x \in J$, let $dF_n(x) = P\{\tau_- > n; U_n \in dx\}$ and let $\Psi(x)$ be the cumulative distribution function (cdf) of a log-increment X_i and $\hat{\Psi}$ its characteristic function. Then the characteristic functions $\widehat{F}_1, \widehat{F}_2, \ldots, \widehat{F}_N$ can be determined recursively through the following relations:

$$\widehat{F}_1 = \widehat{\Psi} \tag{4}$$

$$\widehat{F}_n = \frac{1}{2}\widehat{F}_{n-1} \cdot \widehat{\Psi} + \frac{i}{2}\mathcal{H}\left(\widehat{F}_{n-1} \cdot \widehat{\Psi}\right) \quad \text{for } 2 \leq n \leq N \tag{5}$$

Proof. A straightforward generalization of the recursion on density functions (10) in Ait-

Sahlia and Lai (1998) can be expressed as

$$\begin{aligned} F_1(x) &= \Psi(x) \\ F_n(x) &= 1_J(x) \cdot (F_{n-1} * \Psi)(x), \quad \text{for } 2 \le n \le N \end{aligned}$$

We now recall the following property that relates Fourier and Hilbert transforms for a function ϕ on $\mathbb{R}$ (cf. Stenger (1993) and Feng and Linetsky (2008)):

$$\mathcal{F}\left(\mathbf{1}_{(0,\infty)} \cdot \phi\right)(\xi) = \frac{1}{2}\hat{\phi} + \frac{i}{2}\mathcal{H}\left(\hat{\phi}\right)(\xi),$$

which together with the independence of the Lévy increments leads, for $2 \le n \le N$, to:

$$\begin{aligned} \mathcal{F}(F_n) &= \mathcal{F}\left(1_J \cdot (F_{n-1} * \Psi)\right) \\ &= \frac{1}{2}\mathcal{F}\left(F_{n-1} * \Psi\right) + \frac{i}{2}\mathcal{H}\left(\mathcal{F}\left(F_{n-1} * \Psi\right)\right) \\ &= \frac{1}{2}\widehat{F}_{n-1} \cdot \widehat{\Psi} + \frac{i}{2}\mathcal{H}\left(\widehat{F}_{n-1} \cdot \widehat{\Psi}\right). \quad \square \end{aligned}$$

<u>Remarks</u>

1. The preceding applies to the distribution of the *minimum* of the random walk as well. Simply replace U_n by $-U_n$. Then

$$\Lambda_N = \min\left\{U_n : 0 \le n \le N\right\} = -\max\left\{-U_n : 0 \le n \le N\right\}$$

And for $x < 0$,

$$\begin{aligned} P\left\{\Lambda_N \in dx\right\} &= P\left\{U_1 \in dx\right\} P\left\{\tau_- > N-1\right\} \\ &\quad + \sum_{\nu=2}^{N} P\left\{\tau_+ > \nu; U_\nu \in dx\right\} P\left\{\tau_- > N-\nu\right\} \end{aligned}$$

2. The recursions (2)–(3) fit perfectly the set-up of Feng and Linetsky (2008) to apply their highly efficient algorithm to compute all the Fourier and Hilbert transforms and invert the last ($\widehat{F}_N$)for pricing purposes at a computational cost of $O\left(NM\log(M)\right)$, where

M is the number of quadrature points in the integrals and N is the number of discrete observation dates, with a resulting error $O\left(M^{1/(1+\nu)}\exp(-cM^{\nu/(1+\nu)}\right)$, $c > 0$, which decays exponentially. The ultimate determination of $\widehat{F}_N$ (via its Fourier inversion) is at the root of the computation of the option price as we show next.

5 Fixed-strike lookback options

We are now ready to apply the main result of the last section to price a fixed strike (a.k.a. hindsight) lookback option, which, upon exercise, grants the right to purchase the underlying asset at the minimum price and re-sell it at the strike K, for a put, or to buy it at the strike K and re-sell it at the maximum for a call. To enable comparisons with earlier results involving only Brownian motion, we shall focus on the call, whose payoff is $\left(S_0 e^{M_N} - K\right)^+$.

Proposition 2. The value of a hindsight (or fixed-strike) lookback call at inception is

$$e^{-rT} E\left(S_0 e^{M_N} - K\right)^+ = e^{-rT}\alpha_N\left(S_0 - K\right)^+ + e^{-rT}\sum_{\nu=1}^{N}\int_0^\infty \left(S_0 e^x - K\right)^+ dF_\nu(x), \quad (4)$$

where $F_\nu(x)$ are obtained through the application of the numerical scheme of Feng and Linetsky (2008) to the recursions (2)–(3) for $x > 0$, with $J = (-\infty, 0]$, and $\alpha_0, \alpha_1, \ldots, \alpha_N$ defined by

$$\alpha_0 = 1, \quad \alpha_n = G_n(0) - \lim_{x\to-\infty} G_n(x) \ \text{ for } n \geq 1,$$

where G_n defined for $x \leq 0$ by replacing $\widehat{F}_n$ by $\widehat{G}_n$ in (2)–(3) and using $J = (-\infty, 0]$.

Proof. By definition, we have

$$E\left(S_0 e^{M_N} - K\right)^+ = \int_0^\infty \left(S_0 e^x - K\right)^+ P\left\{M_N \in dx\right\},$$

the right hand side of whcih can be re-expressed as

$$(S_0 - K)^+ P\{M_N = 0\} + \int_{0+}^{\infty} (S_0 e^x - K)^+ P\{M_N \in dx\}.$$

Recall that $\tau_+ = \inf\{n : U_n > 0\}$ and $dG_n(x) = P\{\tau_+ > n; U_n \in dx\}$ for $x < 0$ and $n \geq 1$. Therefore

$$\alpha_n = \int_{-\infty}^{0} dG_n(x) = P\{\tau_+ > n\} = P\{U_1 \leq 0, \ldots, U_n \leq 0\}. \tag{5}$$

The latter, together with (3) and the decomposition above, yields

$$P\{M_N \in dx\} = \alpha_{N-1} P\{U_1 \in dx\} + \sum_{\nu=2}^{N} \alpha_{N-\nu} dF_\nu(x) \text{ for } x > 0,$$

which in turn concludes the proof by virtue of $P\{M_N = 0\} = P\{\tau_+ > N\}$.

6 Floating-strike lookback options

We show in this section that the pricing via the recursions (2)–(3) extends to floating-strike lookback options. These are contrasted to the fixed-strike by making the strike set to the price of the underlying upon exercise. Thus with a floating-strike put, its holder can purchase the underlying at its trading price upon exercise and sell it at the maximum it has achieved over the life of the contract, resulting in a payoff $\left(S_0 e^{M_N} - S_M\right)^+$. On the other hand, a floating-strike call allows its holder to purchase the asset at the minimum it achieved during its life and sell it at the price it trades upon exercise. Again, to allow for comparison with the calssical Brownian process in the Black-Scholes model we illustrate the application of the approach on the put. Incidentally, floating-strike options are sometimes labelled standard.

Proposition 3. The value at inception of floating-strike lookback put is given by

$$e^{-rT} E\left(S_0 e^{M_N} - S_N\right)^+ = e^{-rT} S_0 \sum_{\nu=0}^{N-1} \beta_{N-\nu} I_\nu,$$

where

$$
\begin{aligned}
\beta_{N-\nu} &= \int_{-\infty}^{0} (1-e^x)\, dG_{N-\nu}(x) \quad \text{for } 0 \le \nu \le N, \\
I_0 &= 1, \;\; I_\nu = \int_0^\infty e^x dF_\nu(x) \quad \text{for } \nu \ge 1,
\end{aligned}
$$

with $\widehat{F}_\nu$ and $\widehat{G}_\nu$ obtained through the recursions (2)–(3) as in Proposition 2.

Proof. Since $S_N = S_0 e^{U_N}$, we have $\left(S_0 e^{M_N} - S_N\right)^+ = S_0\left(e^{M_N} - e^{U_N}\right)^+$, from which

$$
\begin{aligned}
E\left(e^{M_N} - e^{U_N}\right)^+ &= E\left(1 - e^{U_N}\right) 1_{\{U_1<0, U_2<0, \ldots, U_N<0\}} \\
&\quad + E\left(e^{U_1} - e^{U_N}\right) 1_{\{U_1>0, U_1>U_2, \ldots, U_1>U_N\}} \\
&\quad + \sum_{\nu=2}^{N-1} E\left(e^{U_\nu} - e^{U_N}\right) 1_{\{0<U_\nu, U_1<U_\nu, \ldots, U_{\nu-1}<U_\nu, U_\nu>U_{\nu+1}, \ldots, U_\nu>U_N\}} \\
&,
\end{aligned}
$$

where each of the above cases corresponds to the maximum being achieved at, respectively, $t_0 = 0$, t_1, or t_ν, $2 \le \nu \le N-1$. Observe that $P\{U_i = U_j\}$ for $i \ne j$. By definition, $\tau_+ = \inf\{n : U_n > 0\}$ and $\tau_- = \inf\{n : U_n \le 0\}$, but since $P\{U_n = 0\} = 0$ for all $n > 0$, we have $\tau_+ = \inf\{n : U_n \ge 0\}$ almost surely. Therefore

$$
\begin{aligned}
E\left(1 - e^{U_N}\right) 1_{\{U_1<0, U_2<0, \ldots, U_N<0\}} &= E\left(1 - e^{U_N}\right) 1_{\{\tau_+>0\}} \\
&= \int_{-\infty}^{0} (1 - e^x) dG_N(x).
\end{aligned}
$$

Furthermore, we have

$E\left(e^{U_1}-e^{U_N}\right)1_{\{U_1>0,U_1>U_2,\ldots,U_1>U_N\}}$

$$
\begin{aligned}
&= E\left(e^{U_1}-e^{U_1+\sum_{i=2}^{N}X_i}\right)1_{\{U_1>0,U_2-U_1<0,\ldots,U_N-U_1<0\}}\\
&= \int_{x=0}^{\infty}\int_{y=-\infty}^{0}\left(e^x-e^{x+y}\right)\\
&\quad \times P\left\{U_1\in dx, X_2<0, X_2+X_3<0,\ldots,X_2+X_3+\cdots+X_N<0,\sum_{i=2}^{N}X_i\in dx\right\}\\
&= \int_0^{\infty}e^xP\{U_1\in dx\}\\
&\quad \times\left[\int_{-\infty}^{0}(1-e^y)P\left\{X_2<0,X_2+X_3<0,\ldots,X_2+X_3+\cdots+X_N<0,\sum_{i=2}^{N}X_i\in dx\right\}\right]\\
&= \int_0^{\infty}e^x d\Psi(x)\left[\int_{-\infty}^{0}(1-e^y)dG_{N-1}(y)\right],
\end{aligned}
$$

where we make use of the independence between U_1 and $(X_2,\ldots,X_N)$ in the next to last step above.

Finally,

$\sum_{\nu=2}^{N-1}E\left(e^{U_\nu}-e^{U_N}\right)1_{\{0<U_\nu,U_1<U_\nu,\ldots,U_{\nu-1}<U_\nu,U_\nu>U_{\nu+1},\ldots,U_\nu>U_N\}}$

$$
\begin{aligned}
&= \sum_{\nu=2}^{N-1}\int_{x=0}^{\infty}\int_{y=-\infty}^{0}\left(e^x-e^{x+y}\right)P\{U_1<U_\nu,\ldots,U_{\nu-1}<U_\nu;U_n\in dx\}\\
&\quad \times P\{X_{\nu+1}<0,\ldots,X_{\nu+1}+\cdots+X_N<0;X_{\nu+1}+\cdots+X_N\in dy\}\\
&= \sum_{\nu=2}^{N-1}\int_0^{\infty}e^x dF_\nu(x)\left[\int_{-\infty}^{0}(1-e^y)dG_{N-\nu}(y)\right]\quad\square.
\end{aligned}
$$

7 Extensions

Further applications of the technique presented above can be made with straightforward modifications to situations where the payoff depends on the minimum. In addition, all these options can be valued at other times than their inceptions by conditioning on the suprema up to the valuation time prior to expiration. Other variations on the pricing of these lookback include the situation, for example, where the suprema are observed over a predefined window within the life of the contract. In all these cases, the general relations provided by AitSahlia and Lai (1998) also apply here, with obvious modifications and will therefore not be repeated here.

Additionally, our approach is particularly well-suited for the computation of hedging parameters, which are especially crucial to the option writer's risk management practice. For example, the fixed-strike lookback price at time 0 of Proposition 2 can be re-written as

$$
\begin{aligned}
e^{-rT}E\left(S_0e^{M_N}-K\right)^+ &= e^{-rT}\alpha_N\left(S_0-K\right)^+ + e^{-rT}\sum_{\nu=1}^{N}\int_0^{\infty}\left(S_0e^x-K\right)^+dF_\nu(x) \\
&= \begin{cases} e^{-rT}\sum_{\nu=1}^{N}\int_{\log(K/S_0)}^{\infty}\left(S_0e^x-K\right))dF_\nu(x) & \text{if } S_0\leq K \\ e^{-rT}\alpha_N\left(S_0-K\right)+e^{-rT}\sum_{\nu=1}^{N}\int_0^{\infty}\left(S_0e^x-K\right)dF_\nu(x) & \text{if } S_0>K \end{cases}
\end{aligned}
$$

from which the delta and gamma parameters (first and second derivatives with respect to S_0, respectively) can easily be computed.

8 Summary

In this paper we extended a recursive algorithm that was originally developed for lookback option pricing when the underlying asset follows a geometric Brownian motion and is monitored at discrete dates within the life of the contract. Our extension to the geometric Lévy processes exploited the duality property of random walks through the use of ladder epochs resulting in recursion expressions for characteristic functions of the extrema that are perfectly tailored for a powerful algorithm for Hilbert transform akin to the Fast Fourier Transform. In addition, our approach yields hedging parameters with little additional computational effort. The ability to develop such results is inherently linked to the characterization of Lévy processes as consisting of continuous-time processes with independent and identically distributed increments. Thus their discrete monitoring is in fact very helpful as it enables us to use readily available results from fluctuation theory.

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