A canonical optimal stopping problem for American options and its numerical solution

Farid AitSahlia

Financial Engines, 1804 Embarcadero Road, Palo Alto, California 94303, USA

Tze Leung Lai

Department of Statistics, Stanford University, Stanford, California 94305, USA

In this paper, the authors present a simple and accurate method for computing the values and early exercise boundaries of American options. A key idea underlying the method is the reduction of American option valuation to a *single* optimal stopping problem for standard Brownian motion indexed by one parameter in the absence of dividends and by two parameters in the presence of a dividend rate. Numerical results obtained by this method show that, in the canonical scale, the stopping boundaries are well approximated by certain piecewise linear functions that can easily be tabulated, leading to new approximations for American option values and hedge parameters.

1. INTRODUCTION

Except for McKean's (1965) formula for perpetual contracts and Merton's (1973) result that American calls written on non-dividend-paying stocks reduce to European calls, American option valuation lacks explicit formulas and has to be performed numerically. There are four approaches to the valuation of standard American options in the literature. The first is based on binomial trees, introduced by Cox, Ross, and Rubinstein (1979), to approximate the underlying geometric Brownian motion (GBM) price process by a discrete-time process on a tree to which dynamic programming can be applied to solve the optimal stopping problem associated with this valuation. The second approach, introduced by Brennan and Schwartz (1977), uses finite difference methods to solve the free boundary partial differential equation defining the function that expresses the option value in terms of time to maturity and the stock price. The third approach uses Monte Carlo simulation; see Boyle, Broadie, and Glasserman (1997) for a recent review. The fourth approach uses analytic approximations to reduce the computational task; see Ingersoll (1998), Ju (1998), and AitSahlia and Lai (1999) for recent reviews of various approximations proposed.

The accuracy of these approximations to American option prices is assessed by comparison with benchmark values obtained by the binomial tree method (with 10 000 or more steps). The binomial tree is constructed with the current stock price as the root node. For different spot prices and maturities, different trees have to be constructed. Moreover, a particular tree stemming from a given spot price as the root node may only have a few exercise nodes, which therefore provide little information on the optimal exercise boundary for the underlying GBM price process. In this paper, we propose an alternative approach which uses a Bernoulli random walk to approximate Brownian motion after a space–time transformation of the option pricing problem, thereby removing the dependence on a prescribed root node in the binomial tree approach. It computes not only the option price by a simple (backward) recursion, as in the binomial tree method, but also the entire early exercise boundary after a continuity adjustment to correct for the difference between the continuous underlying Brownian motion and the discrete approximating Bernoulli random walk.

The basic space–time transformation in our approach also has other important advantages. It reduces all American option valuation problems to a single *canonical optimal stopping problem* indexed by one parameter in the absence of dividends and by two parameters when there is an additional dividend rate. It also multiplies calendar time by σ^2 to transform it into 'canonical' time, where σ is the volatility. This means that for values of σ (between .1 and .4) that are of practical interest, the time horizon $\sigma^2 T$ in the canonical scale is only a small fraction of T.

Section 2 describes the change of variables and the canonical optimal stopping problem. Section 3 describes the Bernoulli walk approach for solving the optimal stopping problem. Section 4 presents numerical results obtained by applying this approach for a wide range of commonly used parameter values in the case of American puts on non-dividend-paying stocks. These results show that early exercise boundaries are well approximated, in the canonical space–time scale, by linear splines with a few knots. They suggest two fast and accurate approximations to American option prices and hedge parameters, presented in Section 5. Section 6 gives similar results for put and call options on dividend-paying stocks. Finally, Section 7 summarizes and concludes the paper.

2. A CANONICAL OPTIMAL STOPPING PROBLEM

In the standard Black–Scholes environment, the price of an American option is the value in the optimal stopping problem $\sup_{\tau \in \mathcal{T}_{0,T}} \mathrm{E}[\mathrm{e}^{-r\tau} f(P_\tau)]$, where, for a given strike price K, $f(P) = (K - P)^+$ or $(P - K)^+$ for a put or a call,

$$P_t = P_0 \mathrm{e}^{(r-\mu-\frac{1}{2}\sigma^2)t+\sigma W_t}, \tag{1}$$

P_0 is the initial security price, $\{W_t\}$ is a standard Brownian motion (so that the stochastic process $\{P_t\}$ is a GBM), and $\mathcal{T}_{a,b}$ is the set of stopping times taking values between a and b with $b > a$. In (1), r is the riskless rate of return, μ stands for the dividend rate paid by the underlying security, and σ is the standard deviation of the security's return. Using arbitrage arguments, Karatzas (1988) has shown that the American option price $U(t, P_t)$ at time $t \in [0, T]$ is given by

$$U(t, P) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathsf{E}[\mathrm{e}^{-r(\tau - t)} f(P_\tau) \mid P_t = P], \tag{2}$$

where the expectation is taken with respect to the risk-neutral measure.

The stopping (early exercise) region $\mathcal{E}$ for the optimal stopping problem (2) has the following characterization:

$$\begin{aligned} U(t, P) &= f(P) \quad \text{if } (t, P) \in \mathcal{E}, \\ U(t, P) &> f(P) \quad \text{if } (t, P) \notin \mathcal{E}. \end{aligned}$$

It is known that $\mathcal{E}$ is a closed subset of $[0, T] \times \mathbb{R}$ consisting of all (t, P) with $0 \leqslant t \leqslant T$ such that $P \leqslant \bar{P}(t)$ for puts and $P \geqslant \bar{P}(t)$ for calls, where $\bar{P}$ is a monotone continuously differentiable function, except for the case of a non-dividend-paying call ($\mu = 0$) whose optimal exercise time is T (see Merton 1973, Van Moerbeke 1976, Karatzas 1988, Jacka 1991). Therefore, when we refer to the American call option in the following, we implicitly assume $\mu > 0$. An analytic reformulation of the optimal stopping problem (2) is provided by the following variational inequality: find $\mathcal{E}$ and U satisfying

$$\begin{aligned} (t, P) \in \mathcal{E} \quad &\Rightarrow \quad U(t, P) = f(P), \\ (t, P) \notin \mathcal{E} \quad &\Rightarrow \quad \begin{cases} (t, P) > f(P), \\ -\dfrac{\partial U}{\partial t} + rU - (r - \mu) P \dfrac{\partial U}{\partial P} - \tfrac{1}{2}\sigma^2 P^2 \dfrac{\partial^2 U}{\partial P^2} = 0. \end{cases} \end{aligned} \tag{3}$$

We now reduce the number of parameters (K, T, r, μ, σ) in the optimal stopping problem (2) by certain space–time transformations. First, by dividing all prices by K, we can reduce the problem to the case $K = 1$, which we shall assume throughout what follows. Next, we introduce the change of variables

$$\begin{aligned} t' &= t - T, \\ y &= \frac{1}{\sigma} \ln \frac{P}{\mathrm{e}^{(r - \mu - \frac{1}{2}\sigma^2)t'}}, \\ u(t', y) &= \mathrm{e}^{-rt'} U\big(t(t'), P(t', y)\big). \end{aligned} \tag{4}$$

In this system of coordinates, (3) can be restated as

$$\begin{aligned} (t', y) \in S \quad &\Rightarrow \quad u(t', y) = h(t', y), \\ (t', y) \notin S \quad &\Rightarrow \quad \begin{cases} u(t', y) > h(t', y), \\ \dfrac{\partial u}{\partial t'} + \tfrac{1}{2} \dfrac{\partial^2 u}{\partial y^2} = 0, \end{cases} \end{aligned}$$

where

$$h(t', y) = \mathrm{e}^{-rt'}(1 - \mathrm{e}^{(r-\mu-\frac{1}{2}\sigma^2)t'+\sigma y})^+$$

in the case of a put,

$$h(t', y) = \mathrm{e}^{-rt'}(\mathrm{e}^{(r-\mu-\frac{1}{2}\sigma^2)t'+\sigma y} - 1)^+$$

in the case of a call (with $\mu > 0$), and S represents the stopping set in the new coordinate system. Since $\partial/\partial t + \frac{1}{2}\partial^2/\partial y^2$ is the infinitesimal generator of space–time Brownian motion, we have therefore expressed the valuation of an American option in terms of the following optimal stopping problem for Brownian motion:

$$u(t', y) = \sup_{\tau \in \mathcal{T}_{t',0}} \mathsf{E}[h(\tau, W_\tau)], \qquad W_{t'} = y. \tag{5}$$

To further reduce the number of parameters, let $s = \sigma^2 t'$ and $z = \sigma y$, so that

$$\begin{aligned} s &= \sigma^2(t - T), \\ z &= \ln(P/K) - (\rho - \alpha\rho - \tfrac{1}{2})s. \end{aligned} \tag{6}$$

Since $\{\sigma^{-1} W_{\sigma^2 t'}\}$ is also a standard Brownian motion, we have the following reduction of American option pricing problems.

THEOREM 1. *Let $\rho = r/\sigma^2$ and $\alpha = \mu/r$. The value of an American option can be obtained from the value function*

$$w(s, z) = \sup_{\tau \in \mathcal{T}_{s,0}} \mathsf{E}[g(\tau, W_\tau)], \qquad W_s = z, \tag{7}$$

where $\{W_s\}$ is a standard Brownian motion, $g(s, z) = \mathrm{e}^{-\rho s}(\mathrm{e}^{[\rho(1-\alpha)-\frac{1}{2}]s+z} - 1)^+$ for a call, and $g(s, z) = \mathrm{e}^{-\rho s}(1 - \mathrm{e}^{[\rho(1-\alpha)-\frac{1}{2}]s+z})^+$ for a put.

Note that for problem (7) the horizon is always 0, and therefore, for a given set of parameters (ρ, α), only one numerical program need be implemented for all expiration dates T. Also, from (6), $s = -\sigma^2 T$ at time $t = 0$. Hence, for values of σ (between .1 and .4) and T (between .08 and 1.5) that commonly arise in practice, $\sigma^2 T$ is typically small (not exceeding .3). Finally, a solution to (7) is a pair $(w, \bar{z})$, where w, the value function, is defined on $(-\infty, 0] \times \mathbb{R}$, and $\bar{z}$, the optimal stopping boundary, is defined on $(-\infty, 0]$. Thus, we retrieve the solution $(U, \bar{P})$ for the original pricing problem by mapping back as follows: $U(t, P) = K\mathrm{e}^{\rho s} w(s, z)$ and $\bar{P}(t) = K\mathrm{e}^{\bar{z}(s)+(\rho-\alpha\rho-\frac{1}{2})s}$.

It is known (see e.g. Kim 1990) that, as $t \to T$,

$$\bar{P}(t) \to \begin{cases} K & \text{if } r \geqslant \mu, \\ (r/\mu)K & \text{if } r < \mu, \end{cases}$$

in the case of a put, while

$$\bar{P}(t) \to \begin{cases} K & \text{if } r \leqslant \mu, \\ (r/\mu)K & \text{if } r > \mu, \end{cases}$$

in the case of a call. In the canonical scale (6), these limits translate to $\lim_{s\to 0}\bar{z}(s) = \kappa_P$ for a put and $\lim_{s\to 0}\bar{z}(s) = \kappa_C$ for a call, where

$$\kappa_P = \begin{cases} 0 & \text{if } 0 \leqslant \alpha \leqslant 1, \\ -\ln\alpha & \text{if } \alpha > 1, \end{cases} \tag{8}$$

$$\kappa_C = \begin{cases} 0 & \text{if } \alpha \geqslant 1, \\ -\ln\alpha & \text{if } 0 < \alpha < 1. \end{cases} \tag{9}$$

For perpetual options (corresponding to the case $T = \infty$), McKean (1965) has shown that $\bar{P}(t)$ is constant and equal to $\theta K/(\theta - 1)$ for a put and $\beta K/(\beta - 1)$ for a call, so we can express McKean's stopping boundary in the canonical scale (6) as

$$z_\ell^P(s) = \ln\left(\frac{\bar{\theta}}{\bar{\theta} - 1}\right) - [\rho(1 - \alpha) - \tfrac{1}{2}]s \tag{10a}$$

for a perpetual put, where $\bar{\theta} = -[\rho(1 - \alpha) - \frac{1}{2}] - \{[\rho(1 - \alpha) - \frac{1}{2}]^2 + 2\rho\}^{1/2}$, and as

$$z_u^C(s) = \ln\left(\frac{\bar{\beta}}{\bar{\beta} - 1}\right) - [\rho(1 - \alpha) - \tfrac{1}{2}]s \tag{10b}$$

for a perpetual call, where $\bar{\beta} = -[\rho(1 - \alpha) - \frac{1}{2}] + \{[\rho(1 - \alpha) - \frac{1}{2}]^2 + 2\rho\}^{1/2}$. The optimal stopping boundary actually lies between (10) and a line parallel to it, as shown by the following theorem.

THEOREM 2 *The stopping boundaries, z^P and z^C, for the canonical optimal stopping problems of the put and the call, respectively satisfy*

$$z_\ell^P(s) \leqslant z^P(s) \leqslant z_u^P(s) \quad \text{and} \quad z_\ell^C(s) \leqslant z^C(s) \leqslant z_u^C(s)$$

for all $s \leqslant 0$, where z_ℓ^P and z_u^C are given by (10a) and (10b), and

$$z_u^P(s) = -(\rho - \alpha\rho - \tfrac{1}{2})s + \kappa_P \quad \text{and} \quad z_\ell^C(s) = -(\rho - \alpha\rho - \tfrac{1}{2})s + \kappa_C,$$

with κ_P and κ_C defined in (8) and (9), respectively.

Proof. With g defined in Theorem 1, let $H(s, z) = (\partial/\partial s + \partial^2/\partial z^2)g(s, z)$. Van Moerbeke (1976) has shown that the optimal stopping boundary belongs to the region $\{(s, z) : H(s, z) \leqslant 0\}$. Since

$$\begin{aligned} H(s, z) &= e^{-\rho s}(\alpha\rho e^{z+(\rho-\alpha\rho-\frac{1}{2})s} - \rho) && \text{for puts,} \\ &= e^{-\rho s}(-\alpha\rho e^{z+(\rho-\alpha\rho-\frac{1}{2})s} + \rho) && \text{for calls,} \end{aligned}$$

it follows that $z^P(s) \leqslant z_u^P(s)$ and $z^C(s) \geqslant z_\ell^C(s)$. In fact, for dividend-paying calls, Section 4.4 of Van Moerbeke (1976) has given these bounds for the optimal stopping boundary. To facilitate reference, Van Moerbeke's parameters (σ, β, δ) are related to those of our canonical problem as follows: $\sigma = 1$, $\beta = \rho$, $\delta = \rho - \alpha\rho - \frac{1}{2}$. □

3. A NUMERICAL METHOD USING BERNOULLI WALKS INSTEAD OF BINOMIAL TREES

The classical binomial tree method is based on using a binomial tree, with root node P_0 at time 0, to approximate the GBM process (1) so that (2) can be computed via a backward induction algorithm to solve the corresponding optimal stopping problem for the approximating binomial tree. The change of variables (6) transforms (2) into the optimal stopping problem (7) for Brownian motion (with $\mathrm{E}[dW_s] = 0$ and $\mathrm{Var}[dW_s] = ds$). In view of the functional central limit theorem, a standard Brownian motion can be approximated by a symmetric Bernoulli random walk, so (7) can likewise be computed via the following backward induction algorithm applied to the approximating random walk.

First choose a small $\delta > 0$ and discretize time and space as follows. Let $s_0 = 0$ and $s_j = s_{j-1} - \delta$ for $j \geqslant 1$. Let

$$\boldsymbol{Z}_\delta = \{\sqrt{\delta}n : n \text{ is an integer}\} = \{0, \pm\sqrt{\delta}, \pm 2\sqrt{\delta}, \dots\}.$$

Approximating Brownian motion by a symmetric Bernoulli random walk with time increment δ and space increment $\sqrt{\delta}X_i$, where the X_i are independent Bernoulli variables with $P(X_i = 1) = \frac{1}{2} = P(X_i = -1)$, we can approximate (7) by the backward recursion

$$w(s_i, z) = \max\{g(s_i, z), \tfrac{1}{2}[w(s_{i-1}, z + \sqrt{\delta}) + w(s_{i-1}, z - \sqrt{\delta})]\}, \tag{11}$$

with $w(s_0, z) = g(0, z)$ and with $z \in \boldsymbol{Z}_\delta$, noting that $s_{i-1} = s_i + \delta$.

Each point z in the lattice $\boldsymbol{Z}_\delta$ can be determined to be a stopping or continuation point at time s_i depending on whether $w(s_i, z) = g(s_i, z)$ or $w(s_i, z) > g(s_i, z)$. Chernoff and Petkau (1986) propose to use the stopping boundary associated with (11) and a continuity correction to compute the optimal stopping boundary in the corresponding problem for Brownian motion. In the case of the canonical optimal stopping problem (7), let $\lfloor x \rfloor = \max\{z \in \boldsymbol{Z}_\delta : z \leqslant x\}$ and $\lceil x \rceil = \min\{z \in \boldsymbol{Z}_\delta : z \geqslant x\}$, and define

$$\begin{aligned} \bar{z}_\delta(s_i) &= \max\{z \in \boldsymbol{Z}_\delta : z \leqslant \lceil z_{\mathrm{u}}(s_i) \rceil + \sqrt{\delta}, w(s_i, z) = g(s_i, z)\} \quad \text{for puts,} \\ &= \min\{z \in \boldsymbol{Z}_\delta : z \geqslant \lfloor z_\ell(s_i) \rfloor - \sqrt{\delta}, w(s_i, z) = g(s_i, z)\} \quad \text{for calls,} \end{aligned} \tag{12}$$

where $z_{\mathrm{u}}(\cdot)$ and $z_\ell(\cdot)$ are the upper and lower bounds for the early exercise boundary in Theorem 2 (i.e. $z_{\mathrm{u}} = z_{\mathrm{u}}^{\mathrm{P}}$ and $z_\ell = z_\ell^{\mathrm{P}}$ for puts). Letting

$$z_\delta^0(s_i) = \bar{z}_\delta(s_i) + \sqrt{\delta}, \quad z_\delta^1(s_i) = \bar{z}_\delta(s_i) + 2\sqrt{\delta} \quad \text{for puts,} \tag{13a}$$

$$z_\delta^0(s_i) = \bar{z}_\delta(s_i) - \sqrt{\delta}, \quad z_\delta^1(s_i) = \bar{z}_\delta(s_i) - 2\sqrt{\delta} \quad \text{for calls,} \tag{13b}$$

$$D_j(s_i) = g(s_i, z_\delta^j(s_i)) - w(s_i, z_\delta^j(s_i)), \qquad \text{for } j = 0, 1, \tag{14}$$

the Chernoff–Petkau continuity correction involves adding to $\bar{z}_\delta(s_i)$ a term that depends on $D_0(s_i)$, $D_1(s_i)$, and $\sqrt{\delta}$. Specifically, the early exercise boundary at s_i can be computed from $\bar{z}_\delta(s_i)$ via this continuity correction as follows:

$$\bar{z}(s_i) = z_\delta^0(s_i) \pm \sqrt{\delta}\,\big|D_1(s_i)/[2D_1(s_i) - 4D_0(s_i)]\big|, \tag{15}$$

where the + and − signs apply to the call and the put, respectively.

Roughly speaking, $z_\delta^0(s_i)$ and $z_\delta^1(s_i)$ are the continuation points in $\boldsymbol{Z}_\delta(s_i)$ that are closest and second closest to the stopping region at s_i, and the adjustment factor in (15) comes from an extrapolation scheme that uses the values of $\Delta_i(z) := g(s_i, z) - w(s_i, z)$ at the two points $z_\delta^0(s_i)$ and $z_\delta^1(s_i)$ to fit a certain functional form of $\Delta_i(z)$ in a neighborhood of $\bar{z}(s_i)$. This functional form is derived from a Taylor expansion of the continuous-time value function $w(s_i, z)$ in (7) about $z = \bar{z}(s_i)$, using the condition of smooth fit at the continuous-time optimal stopping boundary and corresponding results in Chernoff and Petkau (1976) for the approximating Bernoulli walk. Under certain conditions, it can be shown that such continuity corrections can approximate the continuous-time boundary with $o(\sqrt{\delta})$ error; see Chernoff and Petkau (1976) for a special case and Lai, Yao, and AitSahlia (1999) for the general theory.

One such condition underlying (15) is boundedness of the derivative of the optimal stopping boundary in some neighborhood of s_i. In the case of American options, it is known that $\bar{z}(s)$ is continuously differentiable for $s < 0$ and that $\lim_{s\to 0}|\bar{z}'(s)| = \infty$. Numerical results in Section 4 show that the derivative $\bar{z}'(s)$ is large only in a very small neighborhood of $s = 0$, so (15) can be applied when $s_i \leqslant -.005$. For $0 > s > -.005$, $\bar{z}(s)$ is close to κ_C or κ_P, and the uncorrected $\bar{z}_\delta(s)$ typically suffices to approximate this small portion of the early exercise boundary An alternative approach for computing $\bar{z}(s)$ near $s = 0$ is given in AitSahlia and Lai (1999).

We end this section by describing some enhancements of the simple numerical scheme (11). Since (11) only computes $w(s_i, z)$ for $z \in \boldsymbol{Z}_\delta$, the value of $w(s_i, z)$ at $z \notin \boldsymbol{Z}_\delta$ can be computed by interpolation, e.g. by using Lagrange's interpolation formula with a quadratic interpolation polynomial. We can likewise obtain the value of $w(s, z)$ for $s \notin \{s_0, s_1, \dots\}$ by interpolation. Instead of a fixed step size δ, one can take a time-varying step size so that a finer grid is used near $s = 0$ to address the unboundedness of certain derivatives there. This idea is used in Lai, Yao, and AitSahlia (1999) to give a Bernoulli walk scheme, with $O(n)$ time steps, that computes the continuous-time value (7) with $O(n^{-1})$ error. Finally, although (11) can be used to compute $w(s_i, z)$ for all z in the countably infinite set $\boldsymbol{Z}_\delta$, we can restrict z to a finite subset, since in practice there are maximum and minimum values of z at s_n (corresponding to the range of asset prices of interest at an initial date). Moreover, because $w(s, z)$ can be evaluated for any z and s via a closed-form expression given in Sections 5 and 6 once $\bar{z}(\cdot)$ is determined, we can in fact restrict z to the narrow range of values bounded below by $\lfloor z_\ell(s_i)\rfloor - \sqrt{\delta}$ and above by $\lceil z_u(s_i)\rceil + \sqrt{\delta}$, where $z_\ell(\cdot)$ and $z_u(\cdot)$ are the lower and upper bounds for $\bar{z}(\cdot)$ given in Theorem 2.

4. NUMERICAL RESULTS FOR PUTS WITHOUT DIVIDENDS

In this and the next section, we consider puts without dividends so that the canonical stopping problem involves only one parameter ρ. Table 1 illustrates convergence of the Bernoulli walk algorithm (11)–(15) in the case $\rho = 0.5$ as $1/\delta$ varies from 3^6 to 3^{12}, through even powers of 3. Note that if δ' is a positive even power of δ then $\boldsymbol{Z}_\delta$ is a subset of $\boldsymbol{Z}_{\delta'}$. Thus the space–time grids from these values of δ are nested and increase as δ decreases.

Table 1 shows that, for $\delta \leqslant 3^{-8} = 9^{-4}$, the optimal stopping boundary and values $w(s, z)$ of the optimal stopping problem (7) computed by the Bernoulli

TABLE 1. Convergence of stopping boundary $\bar{z}$ and values w of canonical problem for American put with no dividend ($\rho = .5$). The actual put prices in the case $K = 100$, $T = 1$, $\sigma = .2$, $r = .02$ are indicated in parentheses.

		$\delta = 3^{-6}$	$\delta = 3^{-8}$	$\delta = 3^{-10}$	$\delta = 3^{-12}$
$s = -5 \times 3^{-6}$	$\bar{z}(s)$	−0.158403	−0.155849	−0.158798	−0.159090
$(t = 0.8285)$		(85.3506)	(85.5688)	(85.3169)	(85.2920)
	$w(s, -3^{-3})$	0.050865	0.051805	0.051912	0.051924
	$(P = 96.3640)$	(5.0691)	(5.1628)	(5.1734)	(5.1746)
	$w(s, -2 \times 3^{-3})$	0.074905	0.077231	0.077488	0.077517
	$(P = 92.8603)$	(7.4649)	(7.6967)	(7.7223)	(7.7252)
	$w(s, -3 \times 3^{-3})$	0.106228	0.106653	0.106704	0.106710
	$(P = 89.4839)$	(10.5864)	(10.6288)	(10.6339)	(10.6345)
$s = -10 \times 3^{-6}$	$\bar{z}(s)$	−0.214455	−0.205380	−0.205645	−0.206001
$(t = 0.6571)$		(80.6981)	(81.4338)	(81.4122)	(81.3832)
	$w(s, -2 \times 3^{-3})$	0.085042	0.085588	0.085650	0.085656
	$(P = 92.8603)$	(8.4461)	(8.5003)	(8.5065)	(8.5071)
	$w(s, -3 \times 3^{-3})$	0.110081	0.111441	0.111592	0.111609
	$(P = 89.4839)$	(10.9329)	(11.0679)	(11.0829)	(11.0846)
	$w(s, -4 \times 3^{-3})$	0.139470	0.139743	0.139775	0.139779
	$(P = 86.2303)$	(13.8517)	(13.8788)	(13.8820)	(13.8824)
$s = -20 \times 3^{-6}$	$\bar{z}(s)$	−0.261933	−0.263467	−0.262932	−0.263122
$(t = 0.3141)$		(76.9563)	(76.8383)	(76.8794)	(76.8648)
	$w(s, -4 \times 3^{-3})$	0.145948	0.146230	0.146260	0.146264
	$(P = 86.2303)$	(14.3960)	(14.4238)	(14.4267)	(14.4271)
	$w(s, -5 \times 3^{-3})$	0.171994	0.172598	0.172664	0.172671
	$(P = 83.0950)$	(16.9651)	(17.0247)	(17.0312)	(17.0319)
	$w(s, -6 \times 3^{-3})$	0.200048	0.200165	0.200178	0.200180
	$(P = 80.0737)$	(19.7323)	(19.7438)	(19.7451)	(19.7453)

walk method change little with δ, so choosing $\delta = 10^{-4}$ already gives very accurate results. These values can be easily transformed to the American put prices $U(t, P) = Ke^{\rho s}w(s, z)$, where $t = T + s/\sigma^2$ and $P = Ke^{z+(\rho-\frac{1}{2})s}$ is the stock price at time t. The put prices are indicated in parentheses in Table 1 for the case $K = 100$, $T = 1$, $\sigma = .2$, and $r = .02$.

Table 2 tabulates the values of the optimal stopping boundary $\bar{z}(s)$ computed by this method with $\delta = 10^{-4}$, for $s = -.005, -.025, -.05, -.1, -.15, -.3$ and $\rho = .01, .05, .1, .2, \ldots, 2$ in the case of American puts without dividends. Figures 1–3 plot the graphs of the optimal stopping boundaries $\bar{z}(s)$ $(-.3 \leqslant s < 0)$ computed by this method, with $\delta = 10^{-4}$, and the linear splines using the tabulated boundary values in Table 2 as the knots together with $\bar{z}(0) = 0$. They show that the simple method of approximating $\bar{z}(\cdot)$ by splines with these knots performs very well.

TABLE 2. Stopping points for linear spline approximation to boundary of canonical problem for American put with no dividend.

ρ	$\bar{z}(-.3)$	$\bar{z}(-.15)$	$\bar{z}(-.1)$	$\bar{z}(-.05)$	$\bar{z}(-.025)$	$\bar{z}(-.005)$
.01	−1.530588	−1.120455	−0.932000	−0.683088	−0.498143	−0.237556
.05	−1.197890	−0.891489	−0.749844	−0.557773	−0.411207	−0.205308
.10	−1.028656	−0.777312	−0.659271	−0.496108	−0.369880	−0.185140
.20	−0.835710	−0.647515	−0.556691	−0.426663	−0.325206	−0.166178
.30	−0.706895	−0.562916	−0.489372	−0.383111	−0.295858	−0.156741
.40	−0.606378	−0.496936	−0.438313	−0.348249	−0.272120	−0.146239
.50	−0.521933	−0.442398	−0.395862	−0.321070	−0.255259	−0.141607
.60	−0.447562	−0.394697	−0.358964	−0.296966	−0.239270	−0.135396
.70	−0.380792	−0.352156	−0.326139	−0.275899	−0.225348	−0.127679
.80	−0.319320	−0.313554	−0.296563	−0.256741	−0.212299	−0.127439
.90	−0.262027	−0.277920	−0.269074	−0.239304	−0.202086	−0.121504
1	−0.208140	−0.244669	−0.244050	−0.223446	−0.190718	−0.117144
1.1	−0.157054	−0.213016	−0.220032	−0.207763	−0.181983	−0.111053
1.2	−0.108068	−0.183354	−0.197486	−0.193618	−0.172080	−0.108349
1.3	−0.060998	−0.155070	−0.176090	−0.180465	−0.164665	−0.106093
1.4	−0.015861	−0.127575	−0.155609	−0.167414	−0.156013	−0.105708
1.5	0.028401	−0.101236	−0.135910	−0.155427	−0.147847	−0.101974
1.6	0.071100	−0.076061	−0.116882	−0.143741	−0.141021	−0.098839
1.7	0.112693	−0.051142	−0.098457	−0.132057	−0.133406	−0.096296
1.8	0.153390	−0.027388	−0.080619	−0.121446	−0.126707	−0.091879
1.9	0.193317	−0.004191	−0.063458	−0.110661	−0.120510	−0.089812
2	0.232566	0.018789	−0.046685	−0.100645	−0.113679	−0.087918

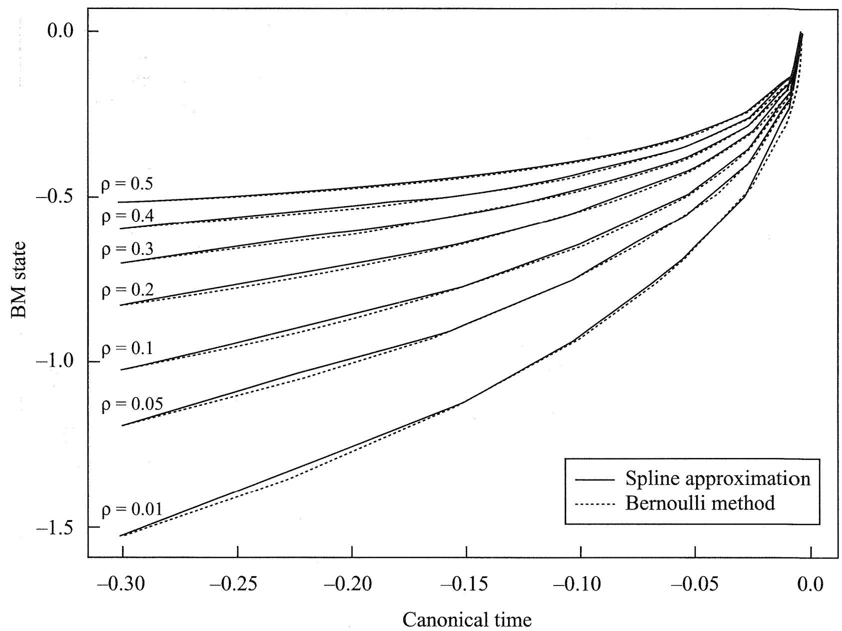

FIGURE 1. Optimal stopping boundaries: put (no dividend) with $\rho \leqslant 0.5$.

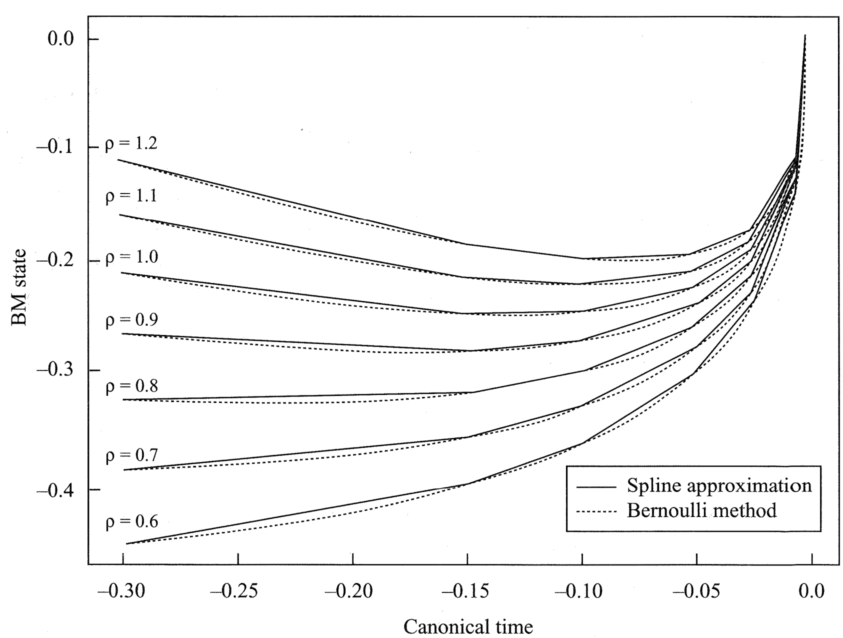

FIGURE 2. Optimal stopping boundaries: put (no dividend) with $0.6 \leqslant \rho \leqslant 1.2$.

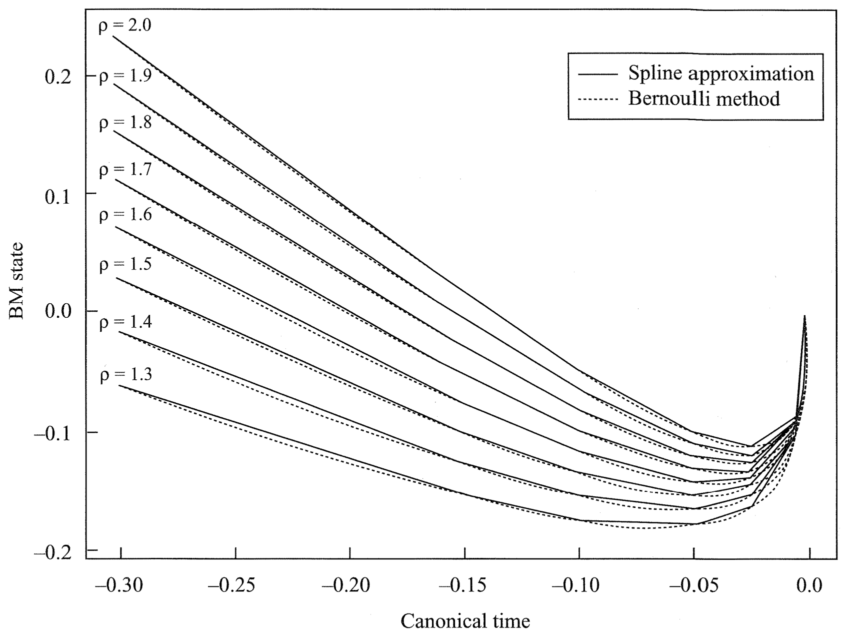

FIGURE 3. Optimal stopping boundaries: put (no dividend) with $1.3 \leqslant \rho \leqslant 2$.

5. EFFICIENT APPROXIMATIONS TO OPTION PRICES AND HEDGE PARAMETERS

The results in Section 4 show that early exercise boundaries for non-dividend-paying American puts are well approximated, in the canonical scale, by linear splines with a few knots. In this section, we show how such approximation to the early exercise boundary can be used to develop fast and accurate approximations to option prices and hedge parameters. Two different approximations are given, and they are both based on this piecewise linear approximation of the early exercise boundary and the decomposition of the American put price $U(t, P)$ due to Kim (1990), Jacka (1991), and Carr, Jarrow, and Myneni (1992):

$$U(t, P) = U_{\mathrm{E}}(t, P) + \int_t^T rK\mathrm{e}^{-r(\tau-t)}\mathrm{N}\big(-d_2(P, \bar{P}(\tau), \tau - t)\big)\,\mathrm{d}\tau,$$

where $\mathrm{N}(\cdot)$ is the standard normal distribution function, U_{E} the Black–Scholes formula for the corresponding European put, and

$$d_2(x, y, \tau) = [\ln(x/y) + (r - \tfrac{1}{2}\sigma^2)\tau]/\sigma\sqrt{\tau}.$$

In the canonical scale, this decomposition of the American option value can be expressed as

$$U(t, P) = Ke^{\rho s}\left[\mathrm{N}\left(-\frac{z}{\sqrt{-s}}\right) - e^{z-\frac{1}{2}s}\mathrm{N}\left(-\frac{z}{\sqrt{-s}} - \sqrt{-s}\right) + \int_s^0 \rho e^{-\rho u}\mathrm{N}\left(\frac{\bar{z}(u) - z}{\sqrt{u-s}}\right) du\right], \tag{16}$$

where s and z are defined by (6) and $\bar{z}(\cdot)$ is the optimal stopping boundary for the canonical problem associated with ρ.

To evaluate the integral in (16), we first note that

$$\rho e^{\rho s}\int_s^0 e^{-\rho u}\mathrm{N}\left(\frac{\bar{z}(u) - z}{\sqrt{u-s}}\right) du = 1 - e^{\rho s} - \int_0^{-s} \rho e^{-\rho t}\mathrm{N}\left(\frac{z - \bar{z}(t+s)}{\sqrt{t}}\right) dt. \tag{17}$$

Suppose that $s = s_m < \cdots < s_0 = 0$ divide the interval $[s, 0]$ into m subintervals such that

$$\bar{z}(u) = \beta_i u + \gamma_i \quad \text{for } s_i \leqslant u \leqslant s_{i-1} \quad (1 \leqslant i \leqslant m). \tag{18}$$

Let $\tau_i = s_i - s_m$. Then, for $1 \leqslant i \leqslant m$, $\bar{z}(t + s_m) - z = -(b_i t + c_i)$ for $\tau_i \leqslant t \leqslant \tau_{i-1}$, where $b_i = -\beta_i$ and $c_i = z - \gamma_i - \beta_i s_m$, noting that $\tau_i + s_m = s_i$. Let $a_i = \sqrt{b_i^2 + 2\rho}$. Then, for $1 \leqslant i \leqslant m$,

$$\begin{aligned}
&\int_{\tau_i}^{\tau_{i-1}} \rho e^{-\rho t}\mathrm{N}\left(\frac{z - \bar{z}(t+s_m)}{\sqrt{t}}\right) dt \\
&\quad = e^{-\rho\tau_i}\mathrm{N}(b_i\tau_i^{1/2} + c_i\tau_i^{-1/2}) - e^{-\rho\tau_{i-1}}\mathrm{N}(b_i\tau_{i-1}^{1/2} + c_i\tau_{i-1}^{-1/2}) \\
&\qquad + \tfrac{1}{2}\left(\frac{b_i}{a_i} + 1\right)e^{(a_i - b_i)c_i}[\mathrm{N}(a_i\tau_{i-1}^{1/2} + c_i\tau_{i-1}^{-1/2}) - \mathrm{N}(a_i\tau_i^{1/2} + c_i\tau_i^{-1/2})] \\
&\qquad + \tfrac{1}{2}\left(\frac{b_i}{a_i} - 1\right)e^{-(a_i + b_i)c_i}[\mathrm{N}(a_i\tau_{i-1}^{1/2} - c_i\tau_{i-1}^{-1/2}) - \mathrm{N}(a_i\tau_i^{1/2} - c_i\tau_i^{-1/2})],
\end{aligned} \tag{19}$$

which corresponds to equation (5) of Ju (1998). We can still apply (19) when $\tau_i = 0$ by replacing τ_i with $t > 0$ and taking the limit as $t \to 0$, which amounts to setting $c_i\tau_i^{-1/2} = 0$, or ∞, or $-\infty$ according as $c_i = 0$, or $c_i > 0$, or $c_i < 0$. Since $\int_0^{-s} = \sum_{i=1}^m \int_{\tau_i}^{\tau_{i-1}}$, the right-hand side integral of (17) can be computed as a sum of terms of the form (19).

To price an American option after the change of variables (6), a simple approximate method is to use Table 2 together with linear interpolation to determine the boundary $\bar{z}(\cdot)$ when ρ belongs to the grid of values in Table 2. Recall that $\bar{z}(0) = 0$. We then use (17) and (19) to compute (16). When ρ falls between two consecutive values ρ_1 and ρ_2 in the grid, we can apply linear interpolation to $\bar{z}_{\rho_1}(\cdot)$ and $\bar{z}_{\rho_2}(\cdot)$ to approximate $\bar{z}_\rho(\cdot)$. Our choice of $\rho = r/\sigma^2$ in Table 2 is based on the range of interest rates $.01 \leqslant r \leqslant .1$ and of volatilities $.1 \leqslant \sigma \leqslant .4$ commonly encountered in practice.

Table 3 contains results from using this approach, labeled SA (spline approximations) to refer to the interpolation spline derived from Table 2, to determine the prices (scaled by K) of puts on non-dividend-paying stocks. The prices in the columns labeled BM (Bernoulli method) result from the application of the dynamic programming algorithm (11) with $\delta = 10^{-4}$. Our results of using this approximation method are within 1% (with an overwhelming majority less than 0.5%) of those obtained by the Bernoulli method.

Formula (16) also leads to explicit expressions for the hedge parameters through differentiation. For example, the parameter delta can be expressed in the canonical scale (6) as

$$\frac{\partial U}{\partial P}(P, t) = -\mathrm{N}\left(-\frac{z}{\sqrt{-s}} - \sqrt{-s}\right) - \rho e^{-z-(\rho-\frac{1}{2})s} \int_s^0 \frac{e^{\rho(s-u)}}{\sqrt{u-s}} \mathrm{n}\left(\frac{\bar{z}(u) - z}{\sqrt{u-s}}\right) du, \quad (20)$$

where $\mathrm{n}(x) = e^{-\frac{1}{2}x^2}/\sqrt{2\pi}$ is the standard normal density function.

TABLE 3. Values of standard American put options (no dividend): BM = Bernoulli method, SA = spline approximation.

		$P/K = .80$		$P/K = .90$		$P/K = 1$		$P/K = 1.2$	
ρ	$-s$	BM	SA	BM	SA	BM	SA	BM	SA
0.125	0.0800	0.22494	0.22503	0.15858	0.15863	0.10777	0.10780	0.04557	0.04558
0.250	0.0133	0.20008	0.19975	0.10870	0.10882	0.04535	0.04458	0.00287	0.00288
0.296	0.1012	0.22492	0.22514	0.16147	0.16164	0.11308	0.11326	0.05243	0.05253
0.305	0.0133	0.19998	0.19959	0.10835	0.10845	0.04424	0.04429	0.00283	0.00284
0.305	0.0533	0.20989	0.21017	0.13762	0.13779	0.08464	0.08483	0.02728	0.02734
0.305	0.0933	0.22224	0.22243	0.15773	0.15787	0.10883	0.10895	0.04852	0.04858
0.305	0.2000	0.24864	0.24878	0.19381	0.19395	0.15033	0.15049	0.08978	0.08988
0.375	0.0133	0.20030	0.19944	0.10785	0.10800	0.04385	0.04393	0.00278	0.00279
0.375	0.0267	0.20152	0.20192	0.11874	0.11907	0.06075	0.06088	0.01055	0.01057
0.375	0.0400	0.20473	0.20495	0.12799	0.12821	0.07310	0.07327	0.01873	0.01880
0.490	0.0612	0.20835	0.20855	0.13714	0.13741	0.08572	0.08598	0.02947	0.02958
0.540	0.0075	0.19996	0.19965	0.10241	0.10266	0.03288	0.03301	0.00053	0.00054
0.540	0.0300	0.20100	0.20128	0.11895	0.11932	0.06208	0.06235	0.01195	0.01197
0.540	0.0525	0.20543	0.20584	0.13174	0.13204	0.07933	0.07954	0.02446	0.02454
0.667	0.0450	0.20245	0.20279	0.12546	0.12591	0.07215	0.07244	0.01951	0.01959
0.800	0.1500	0.21498	0.21551	0.15351	0.15392	0.10914	0.10946	0.05468	0.05486
1.220	0.0033	0.20002	0.20000	0.10000	0.10017	0.02138	0.02151	0.00001	0.00001
1.220	0.0133	0.20002	0.20002	0.10354	0.10377	0.03942	0.03992	0.00219	0.00226
1.220	0.0233	0.20002	0.20075	0.10842	0.10934	0.04979	0.05028	0.00651	0.00660

TABLE 4. Deltas of standard American put options (no dividend): BM = Bernoulli method, SA = spline approximation.

		$P/K = 0.8$		$P/K = 1$		$P/K = 1.2$	
ρ	$-s$	BM	SA	BM	SA	BM	SA
0.250	0.0133	−0.98959	−0.98162	−0.46786	−0.46918	−0.04782	−0.04810
0.375	0.0400	−0.87069	−0.86796	−0.44089	−0.44191	−0.14048	−0.14112
0.490	0.0612	−0.81549	−0.81257	−0.42492	−0.42366	−0.16798	−0.16864
0.540	0.0525	−0.84709	−0.84688	−0.43546	−0.42676	−0.15500	−0.15576
0.667	0.0450	−0.89094	−0.88775	−0.42593	−0.42709	−0.13863	−0.13968
0.800	0.1500	−0.71955	−0.72074	−0.38209	−0.37518	−0.19014	−0.19067
1.220	0.0033	−1.00000	−0.99993	−0.46780	−0.47496	−0.00053	−0.00054
1.220	0.0233	−0.99907	−0.99362	−0.42887	−0.43235	−0.07527	−0.07649

Suppose $\bar{z}(\cdot)$ can be approximated by a piecewise linear function of the form (18) with $s = s_m < \cdots < s_0 = 0$. Then, analogous to (19), the integral in (20) can be evaluated as a sum of m terms:

$$\begin{aligned}\int_{\tau_i}^{\tau_{i-1}} \frac{\mathrm{e}^{-\rho t}}{\sqrt{t}} \mathrm{n}\left(\frac{z - \bar{z}(t + s_m)}{\sqrt{t}}\right) \mathrm{d}t \\ &= a_i^{-1} \mathrm{e}^{(a_i - b_i)c_i} [\mathrm{N}(a_i \tau_{i-1}^{1/2} + c_i \tau_{i-1}^{-1/2}) - \mathrm{N}(a_i \tau_i^{1/2} + c_i \tau_i^{-1/2})] \\ &\quad + a_i^{-1} \mathrm{e}^{-(a_i + b_i)c_i} [\mathrm{N}(a_i \tau_{i-1}^{1/2} - c_i \tau_{i-1}^{-1/2}) - \mathrm{N}(a_i \tau_i^{1/2} - c_i \tau_i^{-1/2})], \quad (21)\end{aligned}$$

with $1 \leqslant i \leqslant m$, as shown in AitSahlia and Lai (1999). Using linear interpolation to determine the linear spline approximation to $\bar{z}(\cdot)$ from Table 2, we can compute the hedge parameter delta via (20) and (21). As shown in Table 4, this approximate method to compute delta yields results that are overwhelmingly within 1% of the benchmark values of the Bernoulli method. The latter are obtained by using in (20) and (21) the optimal stopping boundary $\bar{z}(\cdot)$ that is computed directly by the Bernoulli walk algorithm of Section 3 with $\delta = 10^{-5}$.

The preceding approximation is based on tabulation of the early exercise boundary at a few points that are used as knots for the linear spline approximation to the boundary. If ρ and $\sigma^2 T$ fall outside the range given in Table 2, then we have to use extrapolation, which may be unreliable. An alternative approximation, considered in AitSahlia and Lai (1999), is to set $P = \bar{P}(t)$ $(= K\mathrm{e}^{\bar{z}(s) + (\rho - \frac{1}{2})s})$ in (16) and to assume that $\bar{z}(s)$ is a linear spline in the resultant integral equation for $\bar{z}(s)$, so that $\bar{z}(s_i)$ $(i = 1, \ldots, n)$ can be solved recursively, with $n \leqslant 6$, $\bar{z}(0) = 0$, and $0 = s_0 > s_1 > \cdots > s_n$. Specifically, since

$U(t, \bar{P}(t)) = K - \bar{P}(t)$, (16) yields the integral equation

$$1 - e^{\bar{z}(s)+(\rho-\frac{1}{2})s} = e^{\rho s}\left[N\left(-\bar{z}(s)/\sqrt{-s}\right) - e^{\bar{z}(s)-\frac{1}{2}s}N\left(-\bar{z}(s)/\sqrt{-s} - \sqrt{-s}\right)\right] + \rho e^{\rho s}\int_s^0 e^{-\rho u}N\left(\frac{\bar{z}(u) - \bar{z}(s)}{\sqrt{u - s}}\right) du. \quad (22)$$

Letting $s_n = -\sigma^2 T$, we suppose $\bar{z}(u) = \beta_j u + \gamma_j$ for $s_j \leqslant u \leqslant s_{j-1}$ $(1 \leqslant j \leqslant n)$. Assume $\bar{z}(s_0) = 0, \bar{z}(s_1), \ldots, \bar{z}(s_{m-1})$ have been determined. Then $b_i = -\beta_i = [\bar{z}(s_i) - \bar{z}(s_{i-1})]/(s_{i-1} - s_i)$, $\gamma_i = \bar{z}(s_{i-1}) - \beta_i s_{i-1}$, and $c_i = \bar{z}(s_i) - \bar{z}(s_{i-1}) + \beta_i \tau_{i-1}$ for $1 \leqslant i \leqslant m-1$, where $\tau_j = s_j - s_m$ for $j \leqslant m$ $(\leqslant n \leqslant 6)$. To determine $\bar{z}(s_m)$, let z be a candidate value and let $b(z) = [z - \bar{z}(s_{m-1})]/(s_{m-1} - s_m)$ and $a(z) = [(b^2(z) + 2\rho]^{1/2}$. Noting that $c(z) := z - \bar{z}(s_{m-1}) - (s_{m-1} - s_m)b(z) = 0$ and $\tau_m = 0$, we obtain from (17), (19), and (22) the following equation defining $\bar{z}(s_m)$:

$$\begin{aligned} 1 - e^{z+(\rho-\frac{1}{2})s_m} &= e^{\rho s_m}\left[N\left(-z/\sqrt{-s_m}\right) - e^{z-\frac{1}{2}s_m}N\left(-z/\sqrt{-s_m} - \sqrt{-s_m}\right)\right] \\ &\quad + 1 - e^{\rho s_m} + e^{-\rho\tau_{m-1}}N\left(b(z)\sqrt{\tau_{m-1}}\right) \\ &\quad - \tfrac{1}{2} - \frac{b(z)}{a(z)}N\left(a(z)\sqrt{\tau_{m-1}} - \tfrac{1}{2}\right) - \sum_{i=1}^{m-1} A_i(z), \end{aligned} \quad (23)$$

where $A_i(z)$ is given by the right-hand side of (19) with $c_i = z - \gamma_i - \beta_i s_m$ and b_i and a_i independent of z for $1 \leqslant i \leqslant m-1$. Details of solving the nonlinear equation (23) are given in AitSahlia and Lai (1999), whose numerical results show that this approach is indeed accurate and fast.

6. EXTENSION TO DIVIDEND CASE FOR CALLS AND PUTS

The method in the previous section can be extended to the case where the underlying security pays a dividend at a continuous rate $\mu > 0$. For a put, the integral representation formula generalizing the no-dividend case (16) is

$$\begin{aligned} U(t, P) &= Ke^{\rho s}\left[N\left(-z/\sqrt{-s}\right) - e^{z-\frac{1}{2}s}N\left(-z/\sqrt{-s} - \sqrt{-s}\right)\right] \\ &\quad + \rho K e^{\rho s}\int_s^0 \left[e^{-\rho u}N\left(\frac{\bar{z}(u) - z}{\sqrt{u - s}}\right) - \alpha e^{-\alpha\rho u - \frac{1}{2}s + z}N\left(\frac{\bar{z}(u) - z}{\sqrt{u - s}} - \sqrt{u - s}\right)\right] du, \end{aligned} \quad (24)$$

where z and s are defined by (6) and $\bar{z}(\cdot)$ is the optimal stopping boundary for the canonical optimal stopping boundary associated with the pair (ρ, α). Similarly, the integral representation formula for the value of a call is

$$\begin{aligned} U(t, P) &= Ke^{\rho s}\left[e^{z-\frac{1}{2}s}N\left(z/\sqrt{-s} + \sqrt{-s}\right) - N\left(z/\sqrt{-s}\right)\right] \\ &\quad + \rho K e^{\rho s}\int_s^0 \left[\alpha e^{-\alpha\rho u - \frac{1}{2}s + z}N\left(\frac{z - \bar{z}(u)}{\sqrt{u - s}} + \sqrt{u - s}\right) - e^{-\rho u}N\left(\frac{z - \bar{z}(u)}{\sqrt{u - s}}\right)\right] du. \end{aligned} \quad (25)$$

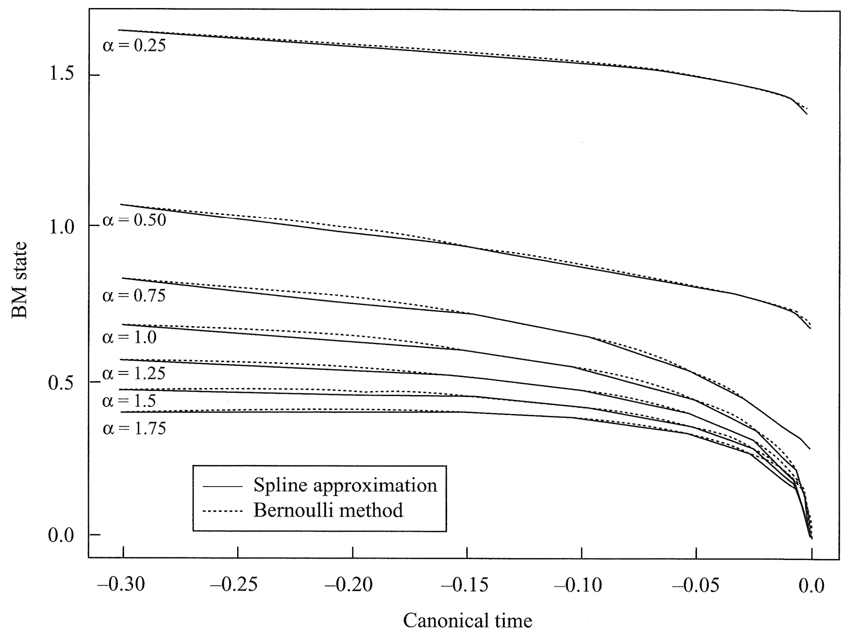

FIGURE 4. Optimal stopping boundaries: call with $\rho = .3$ and $0.25 \leqslant \alpha \leqslant 1.75$.

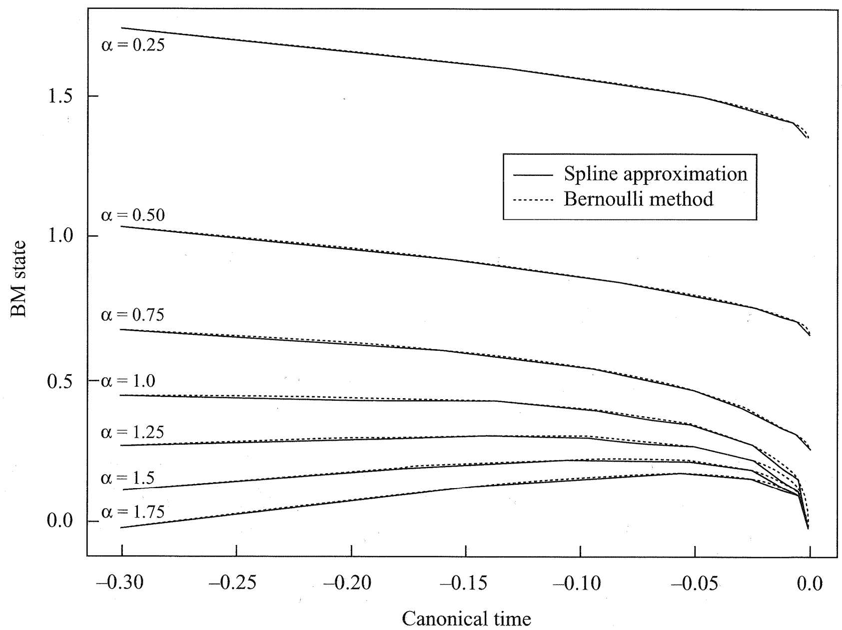

FIGURE 5. Optimal stopping boundaries: call with $\rho = 1.0$ and $0.25 \leqslant \alpha \leqslant 1.75$.

TABLE 5. Stopping points for linear spline approximation to boundary of canonical problem for American put with dividend.

ρ	α	$\bar{z}(-.3)$	$\bar{z}(-.15)$	$\bar{z}(-.1)$	$\bar{z}(-.05)$	$\bar{z}(-.025)$	$\bar{z}(-.005)$
.30	.1	−0.731000	−0.579156	−0.565583	−0.391131	−0.302592	−0.159695
	.5	−0.833955	−0.650903	−0.635428	−0.433340	−0.330881	−0.169023
	.8	−0.918409	−0.713246	−0.695835	−0.473179	−0.363400	−0.186073
	1.0	−0.978708	−0.760057	−0.741265	−0.506429	−0.388270	−0.209430
	1.2	−1.042723	−0.811043	−0.791360	−0.546945	−0.426109	−0.245122
	2.0	−1.332568	−1.083219	−1.063941	−0.872099	−0.812273	−0.742838
	2.5	−1.532323	−1.289682	−1.272500	−1.099409	−1.037210	−0.965732
.5	.1	−0.553328	−0.462951	−0.453833	−0.331195	−0.262744	−0.145912
	.5	−0.687433	−0.552544	−0.540576	−0.381880	−0.297015	−0.159165
	.8	−0.798204	−0.630353	−0.615975	−0.428179	−0.330612	−0.176156
	1.0	−0.877376	−0.688713	−0.672525	−0.467307	−0.364056	−0.195138
	1.2	−0.961071	−0.753562	−0.736039	−0.516600	−0.406123	−0.243607
	2.0	−1.335194	−1.086222	−1.067231	−0.878256	−0.815256	−0.743548
	2.5	−1.578618	−1.317442	−1.298522	−1.109808	−1.043276	−0.966268
1.0	.1	−0.255380	−0.273320	−0.272733	−0.237179	−0.200717	−0.121113
	.5	−0.456121	−0.399623	−0.393531	−0.301762	−0.244413	−0.138202
	.8	−0.622223	−0.508884	−0.498825	−0.362933	−0.286410	−0.158025
	1.0	−0.740979	−0.591157	−0.578330	−0.412720	−0.325584	−0.180600
	1.2	−0.866251	−0.682619	−0.666754	−0.476824	−0.381536	−0.237855
	2.0	−1.407083	−1.124522	−1.103532	−0.895477	−0.824650	−0.746271
	2.5	−1.735322	−1.396338	−1.372557	−1.136365	−1.055949	−0.969154

Closed-form expressions for the integrals in (24) and (25) are also available when $\bar{z}(\cdot)$ is piecewise linear; see Ju (1998) and AitSahlia and Lai (1999).

Similarly to the no-dividend case, the optimal stopping boundaries in the present situation are also well approximated by linear splines, as illustrated in Figures 4 and 5. Tables 5 and 6 provide analogues of Table 2 for the American put and call, respectively, with $\alpha = \mu/r > 0$. We take the dividend rate μ to be between 0.1 and 2.5 times the riskless interest rate. To save space, only excerpts of the tables with $\rho = .3$, .5, and 1 (instead of the full range in Table 2) and seven values of α are provided. Note that $\bar{z}(0)$ need not be 0; its value depends on α and is given by (8) for a put and (9) for a call. Taking the put as an example, Table 7 shows our simple method to yield results that are within 1% (with the majority less than 0.5%) of those obtained by the Bernoulli walk

TABLE 6. Stopping points for linear spline approximation to boundary of canonical problem for American call with dividend.

ρ	α	$\bar{z}(-.3)$	$\bar{z}(-.15)$	$\bar{z}(-.1)$	$\bar{z}(-.05)$	$\bar{z}(-.025)$	$\bar{z}(-.005)$
.30	.1	2.556111	2.504333	2.481250	2.431250	2.405000	2.355000
	.5	1.069128	0.949343	0.938257	0.817987	0.785000	0.736500
	.8	0.794713	0.700041	0.688615	0.523224	0.423128	0.268158
	1.0	0.678919	0.609775	0.601512	0.457010	0.366590	0.201523
	1.2	0.587162	0.543009	0.535774	0.412975	0.329906	0.182954
	2.0	0.330963	0.365567	0.364680	0.311210	0.258844	0.146745
	2.5	0.214049	0.287582	0.289864	0.270149	0.230020	0.137590
.5	.1	2.594286	2.520862	2.485455	2.435667	2.399000	2.348333
	.5	1.038548	0.935418	0.925427	0.820772	0.785706	0.735811
	.8	0.715086	0.644851	0.635491	0.492619	0.405493	0.267299
	1.0	0.577493	0.539177	0.532664	0.417986	0.338350	0.187024
	1.2	0.467880	0.460164	0.456153	0.367941	0.300845	0.166725
	2.0	0.156299	0.249565	0.253545	0.251257	0.218828	0.130240
	2.5	0.009822	0.156367	0.164430	0.204331	0.186563	0.121275
1.0	.1	2.690977	2.569615	2.559717	2.452241	2.406905	2.348333
	.5	1.035177	0.936388	0.927297	0.828374	0.790151	0.737500
	.8	0.622535	0.575816	0.569046	0.456432	0.381583	0.266987
	1.0	0.441086	0.441611	0.438475	0.364053	0.301460	0.175511
	1.2	0.294396	0.339615	0.340138	0.302897	0.255737	0.148322
	2.0	−0.139072	0.061579	0.073650	0.157069	0.156724	0.107843
	2.5	−0.353026	−0.067098	−0.049049	0.095960	0.117490	0.095186

TABLE 7 Values of standard American put options (with dividend): BM = Bernoulli method, SA = spline approximation.

			$P/K = .90$		$P/K = 1$		$P/K = 1.2$	
ρ	α	$-s$	BM	SA	BM	SA	BA	SA
0.3	0.1	0.0133	0.10852	0.10866	0.04444	0.04446	0.00286	0.00286
1.0	0.1	0.0133	0.10486	0.10501	0.04103	0.04130	0.00239	0.00244
1.0	0.1	0.0533	0.12470	0.12525	0.07299	0.07334	0.02151	0.02164
1.0	0.1	0.2000	0.15575	0.15629	0.11329	0.11368	0.06079	0.06100
1.0	0.5	0.0133	0.10681	0.10689	0.04296	0.04309	0.00267	0.00269
1.0	0.5	0.0533	0.13127	0.13160	0.07927	0.07951	0.02476	0.02483
1.0	0.5	0.2000	0.17192	0.17208	0.12973	0.12996	0.07417	0.07433
1.0	1.0	0.0133	0.10970	0.10978	0.04547	0.04549	0.00302	0.00303
1.0	1.0	0.0533	0.14097	0.14112	0.08822	0.08828	0.02945	0.02947
1.0	1.0	0.2000	0.19653	0.19663	0.15424	0.15438	0.09457	0.09465

method, indicating that the quality of our present approximation is similar to that in the no-dividend case.

7. CONCLUSIONS

This paper describes a Bernoulli walk method for computing *both* the value and the early exercise boundary of an American option. The Bernoulli walk is a natural alternative to the binomial tree after we use a change of variables to reduce the optimal stopping problem to its *canonical* form indexed by only one parameter in the absence of dividends and by two parameters when there is an additional dividend rate. The time horizon in the canonical scale is considerably smaller than in the calendar time scale. The Bernoulli walk method also incorporates a continuity correction to compute the early exercise boundary, which is shown by numerical results obtained using this method to be well approximated by a linear spline with a few knots. This approximately piecewise linear shape of the early exercise boundary in the canonical scale suggests two fast and accurate approximate methods for computing the values and hedge parameters of standard American options. The first, studied in detail here, is based on tabulation of the exercise boundary at a few prespecified knots and for a grid of parameter values. The second, treated in detail in AitSahlia and Lai (1999), is based on solving the integral equation for the boundary, assuming that it is a linear spline with these knots. Both methods then use the closed-form expressions (16)–(21) (or their extensions when there are dividends) to compute the option values and hedge parameters once this linear spline approximation is determined. They are particularly useful for the practical management of option books, as the implementation of dynamic hedging strategies rests on rapid computation of a multitude of option prices and hedge parameters daily.

Recently Joubert and Rogers (1997) also proposed a tabulation–interpolation approach to approximate the American option values. However, instead of our tabulation of the early exercise boundaries at a few canonical time points, they tabulate options prices and therefore require much larger tables that have to be stored in a computer as a dictionary. In contrast, our tabulation can be stored in a hand-held calculator which can be used to evaluate the closed-form expressions (16)–(21) and similar expressions when there are dividends.

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