

Random walk duality and the valuation of discrete lookback options

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Use is made of the duality property of random walks to develop a numerical method for the valuation of discrete-time lookback options. This method leads to a recursive numerical integration procedure which is fast, accurate and easy to implement.

Keywords: exotic options, lookback options, recursive numerical integration, random walk duality

1. Introduction

Lookback options are popular in OTC markets for currency hedging. The payoff of a lookback option depends on the minimum or maximum price of the underlying asset over the life of the contract. When the extreme values are continuously monitored, these options can be valued analytically (Conze and Viswanathan, 1991; Goldman *et al.*, 1979a,b). On the other hand, when the maximum or the minimum is only monitored at specific (discrete) dates, mispricing occurs if one uses continuous-time formulas, as illustrated by Broadie *et al.* (1998) and Heynen and Kat (1995), but pricing these options via discrete-time methods presents computational challenges in both speed and accuracy.

In this paper we introduce a new method for the valuation of lookback options where the monitoring dates can be as frequent as daily fixings. This method, based on the duality property of random walks, results in a fast and accurate recursive scheme which requires only univariate numerical integration. The paper is organized as follows. We begin with a brief review of the literature on numerical methods for pricing these options in Section 2. Section 3 provides the details of the procedure. Section 4 illustrates the numerical integration algorithm with a few examples and Section 5 gives some concluding remarks.

2. Literature review

Heynen and Kat (1995) derived pricing formulas for the valuation of discretely monitored lookbacks. These formulas involve multivariate normal integrals. Specifically, if m is the number of price fixings, then one has to evaluate $(m + 1)$ -variate normal distribution functions in these formulas, and Monte

Carlo or quasi-Monte Carlo methods are needed to tackle these integration problems when m is not sufficiently small, say $m \geq 8$ (Heynen and Kat, 1995). An alternative approach is to circumvent the integration task by using a binomial tree to approximate the geometric Brownian motion associated with the underlying security. To implement this approach, Hull and White (1993) proposed a method to keep track of both the asset value and the maximum (minimum) price over the monitoring dates up to the current period. Babbs (1992) and Cheuk and Vorst (1997) developed an alternative backward induction algorithm that involves a time variable and a one-dimensional state variable which is the ratio of the current maximum (minimum) to the current asset price. These binomial-tree-based methods are compared in Kat (1995) with the Monte Carlo method.

Recently, Broadie *et al.* (1998) developed an alternative method in which the discrete-time option price is approximated by the corresponding continuous-time value that is modified by certain correction terms. In order to assess the accuracy of their approximation, they also introduced a trinomial-tree variant of the algorithm proposed by Babbs and by Cheuk and Vorst, and used it as a benchmark to show that their approximation provides accurate results in a number of test cases.

3. Closed-form valuation

Consider the standard Black–Scholes environment where there are two securities: a zero-coupon bond maturing at the expiration date T of the lookback option with a flat term-structure r , and a risky security whose price \tilde{S}_t at time t follows a geometric Brownian motion under the risk-neutral probability measure. Formally,

$$\tilde{S}_t = S_0 e^{B_t} \quad t \geq 0$$

where $B_t = (r - \sigma^2/2) + \sigma W_t$, σ represents the volatility of the return of the underlying security, and $\{W_t\}$ is a standard Brownian motion initialized at 0. When the underlying security of the contract is monitored only at the m dates $\Delta t, \dots, m\Delta t$, with $\Delta t = T/m$, the corresponding discrete-time price model is

$$S_n = S_0 e^{U_n} \quad n = 0, 1, \dots, m \quad (1)$$

where $U_0 = 0$, $U_n = X_1 + X_2 + \dots + X_n$ for $n \geq 1$ and the X_i are independent $N(\mu, \bar{\sigma}^2)$ random variables, with $\mu = (r - \sigma^2/2)\Delta t$, $\bar{\sigma} = \sigma\sqrt{\Delta t}$.

3.1 Duality theory and extrema of random walks

Let X_1, X_2, \dots be independent, identically distributed random variables. Consider the random walk $U_n = X_1 + \dots + X_n$. Let $U_0 = 0$ and

$$\begin{aligned} M_m &= \max\{U_n: 0 \leq n \leq m\} \\ \tau_- &= \inf\{n: U_n \leq 0\} \quad \tau_+ = \inf\{n: U_n > 0\} \end{aligned} \quad (2)$$

Since the X_i are independent and have the same distribution, it follows that for $x > 0$,

$$\begin{aligned}
 P\{M_m \in dx\} &= P\{U_1 \in dx\}P\{X_2 \leq 0, X_2 + X_3 \leq 0, \dots, X_2 + \dots + X_m \leq 0\} \\
 &+ \sum_{v=2}^m P\{U_v > U_i, i < v; U_v \in dx\}P\{X_{v+1} \leq 0, X_{v+1} + X_{v+2} \leq 0, \dots, \\
 &\quad X_{v+1} + \dots + X_m \leq 0\}
 \end{aligned}
 \tag{3}$$

By the duality of random walks (cf. Equation 2.1 on p. 394 of Feller, 1971), for $x > 0$,

$$\begin{aligned}
 P\{U_v > U_i, i < v; U_v \in dx\} &= P\{U_v - U_{v-1} > 0, \dots, U_v - U_1 > 0; U_v \in dx\} \\
 &= P\{U_1 > 0, \dots, U_{v-1} > 0; U_v \in dx\} \\
 &= P\{\tau_- > v; U_v \in dx\}
 \end{aligned}
 \tag{4}$$

On the other hand,

$$\begin{aligned}
 P\{X_{v+1} \leq 0, X_{v+1} + X_{v+2} \leq 0, \dots, X_{v+1} + \dots + X_m \leq 0\} \\
 = P\{U_1 \leq 0, U_2 \leq 0, \dots, U_{m-v} \leq 0\} = P\{\tau_+ > m - v\}
 \end{aligned}
 \tag{5}$$

Putting Equations 4 and 5 into Equation 3 yields

$$P\{M_m \in dx\} = P\{U_1 \in dx\}P\{\tau_+ > m - 1\} + \sum_{v=2}^m P\{\tau_- > v; U_v \in dx\}P\{\tau_+ > m - v\}, x > 0
 \tag{6}$$

Moreover,

$$P\{M_m = 0\} = P\{\tau_+ > m\}
 \tag{7}$$

Thus the distribution of the maximum can be expressed in terms of the so-called ‘ladder epochs’ τ_+ and τ_- . This is particularly useful for pricing discretely monitored options because the quantities in Equations 6 and 7 can be computed recursively, as we shall show in Section 4.

Replacing U_n by $-U_n$ in the preceding argument leads to an analogous representation of the distribution of the minimum

$$\Lambda_m = \min\{U_n: 0 \leq n \leq m\} (= -\max\{-U_n: 0 \leq n \leq m\})
 \tag{8}$$

of the random walk $\{U_n\}$. For $x < 0$,

$$P\{\Lambda_m \in dx\} = P\{U_1 \in dx\}P\{\tau_- > m - 1\} + \sum_{v=2}^m P\{\tau_+ > v; U_v \in dx\}P\{\tau_- > m - v\}
 \tag{9}$$

in analogy with Equation 6.

For the particular case where X_i are independent $N(\mu, \tilde{\sigma}^2)$ random variables so that U_n is the normal random walk in the discrete-time price model (1), the following result (Siegmund, 1985, p. 49) can be used to determine the density function f_v of the measure $P\{\tau_{\pm} > v; U_v \in dx\}$ by recursive numerical integration.

Proposition 1. Let J be either $(0, \infty)$ or $(-\infty, 0]$, and $\tau = \inf\{n: U_n \notin J\}$. For $x \in J$, let $f_n(x)dx = P\{\tau > n; U_n \in dx\}$. Let ϕ denote the density function of the standard normal distribution and let $\psi(x) = \tilde{\sigma}^{-1}\phi((x - \mu)/\tilde{\sigma})$. Then for $x \in J$

$$f_1(x) = \psi(x)$$

$$f_n(x) = \int_J f_{n-1}(y)\psi(x-y)dy \quad \text{for } 2 \leq n \leq m \quad (10)$$

3.2 Fixed strike options

A fixed strike (hindsight) call gives its holder the right to buy the underlying security at a fixed strike price K and to sell it at the maximum price achieved during the life of the option. Correspondingly, a hindsight put grants the right to purchase the underlying security at the minimum price and to sell it at a fixed strike. We shall concentrate on the hindsight call, since the arguments and results for the hindsight put are similar. Define M_m by Equation 2 with the same U_n as in Equation 1. Then the payoff is

$$e^{-rT} E(S_0 e^{M_m} - K)^+$$

where K is the strike price of the call. The main quantity to evaluate, $E(S_0 e^{M_m} - K)^+$, can be expressed in terms of the density functions f_v in Proposition 1.

Proposition 2. The value of a hindsight lookback call at inception is

$$e^{-rT} E(S_0 e^{M_m} - K)^+ = e^{-rT} \alpha_m (S_0 - K)^+ + e^{-rT} \sum_{v=1}^m \alpha_{m-v} \int_0^\infty (S_0 e^x - K)^+ f_v(x) dx \quad (11)$$

where $f_v(x)$ is defined recursively for $x > 0$ by Equation 10 with $J = (0, \infty)$,

$$\alpha_0 = 1 \quad \alpha_k = \int_{-\infty}^0 g_k(x) dx \quad \text{for } k \geq 1$$

in which $g_k(x)$ is the same as the $f_k(x)$ defined for $x \leq 0$ by Equation 10 with $J = (-\infty, 0]$.

Proof. First note that

$$E(S_0 e^{M_m} - K)^+ = \int_0^\infty (S_0 e^x - K)^+ P\{M_m \in dx\}$$

$$= (S_0 - K)^+ P\{M_m = 0\} + \int_{0+}^\infty (S_0 e^x - K)^+ P\{M_m \in dx\} \quad (12)$$

Define τ_+ as in Equation 2. For $n \geq 1$ and $x < 0$, since $g_n(x) dx = P\{\tau_+ > n; U_n \in dx\}$, it follows that

$$\alpha_n = \int_{-\infty}^0 g_n(x) dx = P\{\tau_+ > n\} = P\{U_1 \leq 0, \dots, U_n \leq 0\} \quad (13)$$

By Equations 6 and 13, $P\{M_m \in dx\} = \alpha_{m-1} P\{U_1 \in dx\} + \sum_{v=2}^m \alpha_{m-v} f_v(x) dx$ for $x > 0$, yielding Equation 11 in view of Equations 12 and 7.

With Λ_m defined by Equation 8, the price of a fixed strike put is $e^{-rT} E(K - S_0 e^{\Lambda_m})^+$, which can be evaluated by an obvious modification of Proposition 2.

3.3 Floating strike options

The holder of a floating strike (or standard) lookback put has the right to purchase the underlying security at its price on the exercise date, and to sell it at the maximum price it achieved during the life of the option. Correspondingly, a floating strike call is exercised by purchasing the underlying security at the minimum price it achieved during the life of the option, and selling it at the price on the exercise date. In the Black-Scholes environment, the discrete-time price of a floating strike option at time 0 is $e^{-rT} E(S_0 e^{M_m} - S_m)$, which can be evaluated by making use of the following proposition.

Proposition 3. The value of a standard (floating strike) lookback put at inception is

$$e^{-rT} E(S_0 e^{M_m} - S_m) = e^{-rT} S_0 \sum_{\nu=0}^{m-1} \beta_{m-\nu} I_\nu \quad (14)$$

where

$$\beta_{m-\nu} = \int_{-\infty}^0 (1 - e^x) g_{m-\nu}(x) dx \quad \text{for } 0 \leq \nu \leq m-1$$

$$I_0 = 1, I_\nu = \int_0^\infty e^x f_\nu(x) dx \quad \text{for } \nu \geq 1$$

and f_ν and g_ν are the same as in Proposition 2.

Proof. We can express $E(e^{M_m} - e^{U_m})$ as the sum

$$\begin{aligned} & E(1 - e^{U_m}) 1_{\{U_1 < 0, U_2 < 0, \dots, U_m < 0\}} \\ & + E(e^{U_1} - e^{U_m}) 1_{\{U_1 > 0, U_1 > U_2, \dots, U_1 > U_m\}} \\ & + \sum_{\nu=2}^{m-1} E(e^{U_\nu} - e^{U_m}) 1_{\{0 < U_\nu, U_1 < U_\nu, \dots, U_{\nu-1} < U_\nu, U_\nu > U_{\nu+1}, \dots, U_\nu > U_m\}} \end{aligned} \quad (15)$$

where the first and second terms and the ν th summand correspond to the cases $M_m = 0$, $M_m = U_1$ and $M_m = U_\nu$, respectively, noting that $P\{U_i = U_j\} = 0$ for $i \neq j$. Define τ_+ and τ_- as in Equation 2. Since $P\{U_n = 0\} = 0$ for all n , $\tau_+ = \inf\{n: U_n \geq 0\}$ with probability 1, and therefore

$$E(1 - e^{U_m}) 1_{\{U_1 < 0, \dots, U_m < 0\}} = E(1 - e^{U_m}) 1_{\{\tau_+ > m\}} = \int_{-\infty}^0 (1 - e^x) g_m(x) dx$$

The second term in sum (15) can also be written as

$$\begin{aligned} & E(e^{U_1} - e^{U_m}) 1_{\{U_1 > 0, U_2 - U_1 < 0, \dots, U_m - U_1 < 0\}} \\ & = \int_{x=0}^\infty \int_{y=-\infty}^0 (e^x - e^{x+y}) \end{aligned}$$

$$\begin{aligned}
 & P\left\{U_1 \in dx, X_2 < 0, X_2 + X_3 < 0, \dots, X_2 + X_3 + \dots + X_m < 0, \sum_{i=2}^m X_i \in dy\right\} \\
 &= \int_0^\infty e^x P\{U_1 \in dx\} \\
 & \quad \left[\int_{-\infty}^0 (1 - e^y) P\left\{X_2 < 0, X_2 + X_3 < 0, \dots, X_2 + X_3 + \dots + X_m < 0, \sum_{i=2}^m X_i \in dy\right\} \right]
 \end{aligned}$$

where the last step follows from the independence of U_1 and (X_2, \dots, X_m) . Therefore the second term of sum (15) reduces to

$$\int_0^\infty e^x \psi(x) dx \left[\int_{-\infty}^0 (1 - e^y) g_{m-1}(y) dy \right]$$

Similarly, the last term in sum (15) can be written as

$$\begin{aligned}
 & \sum_{v=2}^{m-1} E(e^{U_v} - e^{U_m}) 1_{\{U_v > 0, U_v - U_1 > 0, \dots, U_v - U_{v-1} > 0, U_v - U_{v+1} > 0, \dots, U_v - U_m > 0\}} \\
 &= \sum_{v=2}^{m-1} \int_{x=0}^\infty \int_{y=-\infty}^0 (e^x - e^{x+y}) P\{U_1 < U_v, \dots, U_{v-1} < U_v; U_v \in dx\} \\
 & \quad P\{X_{v+1} < 0, \dots, X_{v+1} + \dots + X_m < 0; X_{v+1} + \dots + X_m \in dy\} \\
 &= \sum_{v=2}^{m-1} \int_0^\infty e^x f_v(x) dx \left[\int_{-\infty}^0 (1 - e^y) g_{m-v}(y) dy \right].
 \end{aligned}$$

With Λ_m defined as in Equation 8, the price of a floating strike lookback call is $e^{-rT} E(S_m - S_0 e^{\Lambda_m})$, which can be evaluated by an obvious modification of Proposition 3.

3.4 Lookbacks with pre-determined extrema and partial lookbacks

The option given in Proposition 2 or 3 is valued at inception. As time evolves, we can also consider the maximum security price to date in order to value the option for the remainder of its life. If we define $\tilde{S}_+(t) = \max_{0 \leq u \leq t} \tilde{S}_u$, then by the usual risk-neutral argument the continuous-time value of a floating strike lookback put is

$$V(t, \tilde{S}, \tilde{S}_+) = e^{-r(T-t)} E[\max\{\tilde{S}_+, \max_{t \leq u \leq T} \tilde{S}_u\} | \tilde{S}_t = \tilde{S}, \tilde{S}_+(t) = \tilde{S}_+] - \tilde{S}$$

Under the discrete-time price model (1) the corresponding option value is

$$V_m(n, S, S_+) = e^{-r(m-n)\Delta t} E[\max\{S_+, \max_{n \leq j \leq m} S_j\} | S_n = S, S_+(n) = S_+] - S$$

where $S_+(n) = \max\{S_j: 0 \leq j \leq n\}$.

Proposition 4. Let $a = S_+/S$. Then

$$V_m(n, S, S_+) = e^{-rm'\Delta t} \left(S_+ \alpha_{m'} + S \sum_{v=1}^{m'} \alpha_{m'-v} \int_0^\infty \max(a, e^x) f_v(x) dx \right) - S$$

where $m' = m - n$, and α_j and $f_v(x)$ are the same as in Proposition 2.

Proof. Note that

$$\begin{aligned} V_m(n, S, S_+) &= e^{-r(m-n)\Delta t} E[\max\{S_+, Se^{M_{m'}}\}] - S \\ &= e^{-rm'\Delta t} SE \left[\max\left\{ \frac{S_+}{S}, e^{M_{m'}} \right\} \right] - S \end{aligned} \tag{16}$$

where $M_{m'}$ is defined in Equation 2. With $a = S_+/S$,

$$\begin{aligned} E[\max\{a, e^{M_{m'}}\}] &= aP\{M_{m'} = 0\} + \int_{0+}^\infty \max(a, e^x) P\{M_{m'} \in dx\} \\ &= a\alpha_{m'} + \sum_{v=1}^{m'} \alpha_{m'-v} \int_0^\infty \max(a, e^x) f_v(x) dx \end{aligned}$$

where the last expression follows from Equations 6, 7 and 13, with m replaced by m' .

Straightforward modifications of the preceding argument yield obvious analogues of Proposition 4 for

$$\begin{aligned} &S - e^{-r(m-n)\Delta t} E[\min\{S_-, \min_{n \leq j \leq m} S_j\} | S_n = S, S_-(n) = S_-] \\ &e^{-r(m-n)\Delta t} E[(K - \min\{S_-, \min_{n \leq j \leq m} S_j\})^+ | S_n = S, S_-(n) = S_-] \\ &e^{-r(m-n)\Delta t} E[(\max\{S_+, \max_{n \leq j \leq m} S_j\} - K)^+ | S_n = S, S_+(n) = S_+] \end{aligned}$$

which correspond to a discretely monitored floating strike lookback put, hindsight put and hindsight call, respectively, at the n th fixing date, where $S_-(n) = \min\{S_j: 0 \leq j \leq n\}$.

The basic idea behind Proposition 4 can also be applied to the problem of pricing partial lookback options recently studied by Heynen and Kat (1995). Instead of covering the full lifetime of the option, the lookback period is often limited to only part of the option's lifetime. As does discrete monitoring, partial monitoring causes lookback options to become less expensive. In particular, for a discretely monitored floating strike call with a lookback period starting at the inception of the option but ending at time $k\Delta t$ before the expiration date T so that $k < m$, Heynen and Kat (1995) derived a formula for pricing the option at the n th fixing date given the underlying price S_n and the monitored minimum price $S_-(n)$ at that date. The formula expresses

$$E[(S_m - \min\{S_-(n), \min_{n \leq j \leq k} S_j\})^+ | S_n = S, S_-(n) = S_-] \tag{17}$$

in terms of $(k - n + 1)$ -variate normal integrals. The arguments leading to Propositions 3 and 4 also provide an alternative representation which does not involve multivariate normal integrals for Expression 17. For ease of comparison with Propositions 3 and 4, we consider a floating strike lookback put instead of the lookback call in Expression 17. We prove the following analogue of Proposition 4 for a partial lookback put. Let $S_+(n) = \max_{0 \leq j \leq n} S_j$, as in Proposition 4.

Proposition 5. Let $c \geq 0$. Let f_ν and g_ν be the same as in Proposition 2 and define $g_\nu^*(x)$ for $x \leq c$ recursively as follows:

$$g_1^*(x) = \psi(x) \quad g_\nu^*(x) = \int_{-\infty}^c g_{\nu-1}^*(y)\psi(x-y)dy \quad \text{for } \nu \geq 2 \tag{18}$$

Let Φ be the standard normal distribution function (i.e. $\Phi(y) = \int_{-\infty}^y \phi(x)dx$), and define for $j \geq 1$ the functions

$$\begin{aligned} \Psi_j(y) &= \Phi((-y - j\mu)/(\sqrt{j\tilde{\sigma}})) - \exp\{y + j(\mu + \tilde{\sigma}^2)^2/(2\tilde{\sigma}^2)\}\Phi((-y - j(\mu + \tilde{\sigma}^2))/(\sqrt{j\tilde{\sigma}})) \\ \Psi_j^*(y; c) &= \\ &e^c\Phi((c - y - j\mu)/(\sqrt{j\tilde{\sigma}})) - \exp\{y + j(\mu + \tilde{\sigma}^2)^2/(2\tilde{\sigma}^2)\}\Phi((c - y - j(\mu + \tilde{\sigma}^2))/(\sqrt{j\tilde{\sigma}})) \end{aligned}$$

Let $m > k > n$. Then

$$\begin{aligned} &E[(\max\{S_+(n), \max_{n \leq j \leq k} S_j\} - S_m)^+ | S_n = S, S_+(n) = e^c S] \\ &= S \int_{-\infty}^c g_{k-n}^*(y)\Psi_{m-k}^*(y; c)dy + S\Psi_{m-k}(0) \int_c^\infty e^x f_{k-n}(x)dx \\ &\quad + S \sum_{\nu=1}^{k-n-1} \left\{ \int_c^\infty e^x f_\nu(x)dx \right\} \left\{ \int_{-\infty}^0 g_{k-n-\nu}(y)\Psi_{m-k}(y)dy \right\} \end{aligned}$$

Proof: Let $k' = k - n, m' = m - n$. Analogous to Equation 16, we can write

$$\begin{aligned} &E[(\max\{S_+(n), \max_{n \leq j \leq k} S_j\} - S_m)^+ | S_n = S, S_+(n) = e^c S] \\ &= SE[(\max\{e^c, e^{M_{k'}}\} - e^{U_{m'}})^+] \end{aligned} \tag{19}$$

Note that $U_{m'} = U_{k'} + Z$, where $Z = \sum_{i=k'+1}^{m'} X_i$ is normal with mean $(m - k)\mu$ and variance $(m - k)\tilde{\sigma}^2$ and is independent of $(M_{k'}, U_{k'})$. Let $h(z)$ denote the density function of Z . Let $\tau_c = \inf\{n: U_n > c\}$ and note that $\{M_{k'} \leq c\} = \{\tau_c > k'\}$. Analogous to Proposition 1,

$$P\{\tau_c > n; U_n \in dx\} = g_n^*(x)dx \quad \text{for } x < c$$

where the g_n^* are defined recursively by Equation 18. Therefore

$$\begin{aligned} E[(\max\{e^c, e^{M_{k'}}\} - e^{U_{m'}})^+] &= E[(e^{M_{k'}} - e^{U_{k'}+Z})^+ 1_{\{M_{k'} > c\}}] + E[(e^c - e^{U_{k'}+Z})^+ 1_{\{\tau_c > k'\}}] \\ &= \sum_{\nu=1}^{k'} E[(e^{U_\nu} - e^{U_{k'}+Z})^+ 1_{\{M_{k'}=U_\nu > c\}}] \\ &\quad + \int_{-\infty}^c E[(e^c - e^{y+Z})^+ 1_{\{y+Z \leq c\}}] g_{k'}^*(y)dy \end{aligned} \tag{20}$$

Since $h(z) = (m - k)^{-1/2} \tilde{\sigma}^{-1} \phi([z - (m - k)\mu]/[(m - k)^{1/2} \tilde{\sigma}])$, a standard completing-the-squares argument yields

$$\begin{aligned}
 E[(e^c - e^{y+Z})1_{\{y+Z \leq c\}}] &= \int_{-\infty}^{c-y} (e^c - e^{y+z})h(z)dz \\
 &= e^c \int_{-\infty}^{c-y} h(z)dz - e^y \int_{-\infty}^{c-y} e^z h(z)dz \\
 &= \Psi_{m-k}^*(y; c)
 \end{aligned} \tag{21}$$

An argument similar to that in the proof of Proposition 3 can be used to show that for $1 \leq \nu \leq k' - 1$,

$$\begin{aligned}
 &E[(e^{U_\nu} - e^{U_{k'+Z}})^+ 1_{\{U_\nu > U_{\nu-1}, \dots, U_\nu > U_1, U_\nu > c\}} 1_{\{X_{\nu+1} < 0, \dots, X_{\nu+1} + \dots + X_{k'} < 0\}}] \\
 &= \int_c^\infty f_\nu(x) \int_{-\infty}^0 g_{k'-\nu}(y) E[(e^x - e^{x+y+Z})1_{\{y+Z \leq 0\}}] dy dx \\
 &= \int_c^\infty e^x f_\nu(x) \int_{-\infty}^0 g_{k'-\nu}(y) \left\{ \int_{-\infty}^{-y} h(z) dz - e^y \int_{-\infty}^{-y} e^z h(z) dz \right\} dy dx \\
 &= \left\{ \int_c^\infty e^x f_\nu(x) dx \right\} \left\{ \int_{-\infty}^0 g_{k'-\nu}(y) \Psi_{m-k}(y) dy \right\}
 \end{aligned} \tag{22}$$

Similarly, in the case $\nu = k'$,

$$\begin{aligned}
 E[(e^{U_{k'}} - e^{U_{k'+Z}})^+ 1_{\{U_{k'} > U_{k'-1}, \dots, U_{k'} > U_1, U_{k'} > c\}}] &= \int_c^\infty f_{k'}(x) E[(e^x - e^{x+Z})1_{\{Z \leq 0\}}] dx \\
 &= \left\{ \int_c^\infty e^x f_{k'}(x) dx \right\} \left\{ \int_{-\infty}^0 (1 - e^z) h(z) dz \right\} \\
 &= \left\{ \int_c^\infty e^x f_{k'}(x) dx \right\} \Psi_{m-k}(0)
 \end{aligned} \tag{23}$$

Expressions 21, 22 and 23 yields the desired conclusion in view of Equations 19 and 20.

4. Numerical implementation and examples

Propositions 2–5 express the values of the options considered here in terms of integrals of the form

$$\int_0^\infty f_n(x)H(x)dx \quad \int_c^\infty f_n(x)H(x)dx \quad \int_{-\infty}^0 g_n(x)H(x)dx \quad \int_{-\infty}^c g_n^*(x)H(x)dx \tag{24}$$

where $f_n(x)$ (for $x > 0$) and $g_n(x)$ (for $x < 0$) are defined recursively via Equation 10, g_n^* (for $x \leq c$) is defined recursively by Equation 18 and $H(x)$ is a function which is of the order $O(e^{|x|})$ as $|x| \rightarrow \infty$. In this section we describe an algorithm to compute recursively the integrals in Expression 24. There are two major issues concerning these integrals. The first is related to recursive updating of the density functions f_n , g_n and g_n^* . The second is related to the unbounded range of integration. To deal with the second issue, we adopt the following truncation. For $B \geq 4$, define $A_n = \max\{(\mu + \tilde{\sigma}^2)n + B\tilde{\sigma}\sqrt{n}, 1\}$ and $\bar{A}_n = \min\{\mu n - B\tilde{\sigma}\sqrt{n}, -1\}$. The inclusion of 1 (resp. -1) is designed to address the case $\mu < 0$

(resp. $\mu > 0$) for \int_0^∞ (resp. $\int_{-\infty}^0$). Since $f_n(x)dx = P\{\tau_- > n; U_n \in dx\} \leq P\{U_n \in dx\}$ and U_n is normal with mean μn and standard deviation $\tilde{\sigma}\sqrt{n}$, $f_n(x)$ is negligibly small for $x > A_n$ and $\int_{A_n}^\infty e^x f_n(x)dx$ is also negligibly small, so we can regard $f_n(x)$ as 0 if $x \geq A_n$. Indeed, simple algebra shows

$$\int_{A_n}^\infty e^x P\{U_n \in dx\} = e^{n\mu + n\tilde{\sigma}^2/2} (1 - \Phi(C_n)) \tag{25}$$

where $C_n = \{A_n - n(\mu + \tilde{\sigma}^2)\}/(\tilde{\sigma}\sqrt{n})$. Note that $A_n \geq n(\mu + \tilde{\sigma}^2) + B\tilde{\sigma}\sqrt{n}$ and therefore $C_n \geq B$. Hence

$$1 - \Phi(C_n) < \phi(C_n)/C_n \leq \phi(B)/B \tag{26}$$

which is less than 3×10^{-5} since $B \geq 4$. On the other hand, $e^{n\mu + n\tilde{\sigma}^2/2} = e^{nrT/m} \leq e^{rT}$, and for problems of practical interest (T, r) fall mostly in the set $(0, 1] \times [-0.2, 0.2]$ (with negative r , for example, representing a negative difference between domestic and foreign riskless rates in the case of currency lookback options). A similar argument shows that we can likewise treat $g_n(x)$ or $g_n^*(x)$ as 0 if $x \leq \bar{A}_n$.

For the recursive updating of $f_n(x)$, choose some small grid size $\delta > 0$ and let k_n be the smallest integer $\geq A_n/\delta$. Let L be the smallest positive even integer such that $L \geq \max\{1, (\mu + \tilde{\sigma}^2)/\delta\}$. Let

$$y_j = \begin{cases} j\delta & \text{for } j \in \{0, 1, \dots, L\} \text{ and } j\delta \leq y < (j+1)\delta \\ (j+1/2)\delta & \text{for } j \geq L+1 \text{ and } j\delta \leq y < (j+1)\delta \end{cases}$$

Using Simpson's rule for the integral $\int_0^{L\delta}$ and the mid-point rule for the integral $\int_{L\delta}^\infty$, we can compute $f_{n+1}(x)$ via the sum

$$\delta \sum_{j=0}^L w_j f_n(y_j) \psi(x - y_j) + \delta \sum_{j=L+1}^{k_n-1} f_n(y_j) \psi(x - y_j) \tag{27}$$

where $(w_0, w_1, \dots, w_L) = (1/3, 4/3, 2/3, 4/3, 2/3, \dots, 4/3, 2/3, 4/3, 1/3)$, with the pair $(4/3, 2/3)$ repeated $(L/2 - 1)$ times, recalling that L is even. Note that in this recursive algorithm we need only compute $f_{n+1}(x)$ for $x = y_i$ ($i = 0, 1, \dots, k_{n+1} - 1$). Furthermore, after the n th iteration, we need only store the k_n values of y_j and of $f_n(y_j)$ to be used for the next iteration. A similar recursive algorithm can be used for the updating of $g_n(x)$ or $g_n^*(x)$. In particular, to compute $g_{n+1}^*(x) = \int_{-\infty}^c g_n^*(x) \psi(x - y) dy$, let k_n^* be the smallest integer $\geq (c - \bar{A}_n)/\delta$ and let L^* be the smallest even integer such that $-L^* \leq \min\{-1, \mu/\delta\}$. Let

$$y_j^* = \begin{cases} c - j\delta & \text{for } j \in \{0, 1, \dots, L^*\} \text{ and } c - (j+1)\delta < y \leq c - j\delta \\ c - (j+1/2)\delta & \text{for } j \geq L^* + 1 \text{ and } c - (j+1)\delta < y \leq c - j\delta \end{cases}$$

Thus $g_{n+1}^*(x)$ is computed via the sum

$$\delta \sum_{j=0}^{L^*} w_j g_n^*(y_j^*) \psi(x - y_j^*) + \delta \sum_{j=L^*+1}^{k_n^*-1} g_n^*(y_j^*) \psi(x - y_j^*) \tag{28}$$

Such a hybrid Simpson-midpoint rule is convenient not only for updating f_n , g_n and g_n^* but also for computing the integrals in Expressions 24.

Tables 1–4 display numerical results on the above recursive integration algorithm giving an indication of their convergence and accuracy as compared to alternative methods. In Tables 1 and 2, the results in the column labelled ‘Monte Carlo’ are taken from Kat (1995). Each Monte Carlo estimate is based on 10 000 simulation runs, and the ‘width’ given in parentheses refers to half the length of the 95% confidence interval centred at the Monte Carlo estimate. Table 1 also includes Kat’s results on the binomial method of Cheuk and Vorst (1997) with 5200 time steps, while Table 2 also includes his results on Babbs’ (1992) method. In Tables 3 and 4, the results on continuity correction methods are taken from Broadie *et al.* (1998), and the column labelled ‘Trinomial

Table 1. Fixed strike (hindsight) lookbacks.

Option (K)	Cheuk and Vorst	Monte Carlo (width)	Recursive integration ($\delta = 0.02$)	Recursive integration ($\delta = 0.01$)	Recursive integration ($\delta = 0.005$)	Recursive integration ($\delta = 0.0025$)	Recursive integration ($\delta = 0.001$)
Put (95)	4.04	4.25 (0.11)	4.2379	4.2227	4.2272	4.2267	4.2266
Put (100)	7.65	7.65 (0.14)	7.6588	7.6486	7.6480	7.6480	7.6480
Put (105)	12.52	12.52 (0.14)	12.5426	12.5255	12.5246	12.5246	12.5246
Call (95)	15.55	15.55 (0.20)	15.5739	15.5537	15.5526	15.5526	15.5526
Call (100)	10.67	10.67 (0.20)	10.6900	10.6768	10.6761	10.6760	10.6760
Call (105)	6.83	6.99 (0.18)	6.9946	6.9725	6.9772	6.9767	6.9765

$S_0 = 100$, $\sigma = 0.20$, $r = 0.05$, $T = 0.5$
 $m = 13$ (bi-weekly observations)

Table 2. Floating strike lookbacks

Option (m)	Babbs	Monte Carlo (width)	Recursive integration ($\delta = 0.02$)	Recursive integration ($\delta = 0.01$)	Recursive integration ($\delta = 0.005$)	Recursive integration ($\delta = 0.0025$)	Recursive integration ($\delta = 0.001$)
Put (26)	8.81	8.82 (0.14)	8.8884	8.8197	8.8171	8.8170	8.8170
Put (13)	8.21	8.21 (0.14)	8.2181	8.2076	8.2071	8.2070	8.2070
Call (26)	10.62	10.61 (0.19)	10.7018	10.6209	10.6179	10.6177	10.6177
Call (13)	10.12	10.11 (0.19)	10.1300	10.1177	10.1171	10.1170	10.1170

$S_0 = 100$, $\sigma = 0.20$, $r = 0.05$, $T = 0.5$

Table 3. Floating strike lookback put price at inception

m	1st-order correction	2nd order- correction	Recursive integration ($\delta = 0.01$)	Recursive integration ($\delta = 0.005$)	Recursive integration ($\delta = 0.0025$)	Recursive integration ($\delta = 0.001$)	Trinomial method
5	9.1500	10.1820	10.0642	10.0642	10.0642	10.0642	10.0642
10	10.9313	11.4469	11.3978	11.3977	11.3977	11.3977	11.3977
20	12.2084	12.4660	12.4450	12.4446	12.4446	12.4445	12.4446
40	13.1203	13.2491	13.2411	13.2394	13.2393	13.2393	13.2394
80	13.7696	13.8340	13.8370	13.8298	13.8294	13.8294	13.8295
160	14.2310	14.2632	14.2972	14.2626	14.2610	14.2609	14.2610

$S_0 = 100$, $\sigma = 0.30$, $r = 0.10$, $T = 0.5$

Table 4. Floating strike lookback put price with predetermined maximum

m	Trinomial method	Continuity correction	Recursive integration ($\delta = 0.01$)	Recursive integration ($\delta = 0.005$)	Recursive integration ($\delta = 0.0025$)	Recursive integration ($\delta = 0.001$)
5	13.2995	12.7909	13.3010	13.2985	13.2995	13.2994
10	14.1228	13.8557	14.1244	14.1210	14.1220	14.1219
20	14.8060	14.6688	14.8105	14.8042	14.8050	14.8049
40	15.3446	15.2747	15.3609	15.3434	15.3436	15.3434
80	15.7545	15.7190	15.8180	15.7556	15.7533	15.7530
160	16.0591	16.0412	16.3521	16.0704	16.0583	16.0574

$S_0 = 100$, $S_+ = 110$, $\sigma = 0.30$, $r = 0.10$, $T = 0.5$

method' refers to their trinomial-tree adaptation of the methods of Babbs and of Cheuk and Vorst who used binomial trees instead.

Tables 1 and 2 give five choices for the value of δ in the recursive integration algorithm with δ ranging from 0.02 to 0.001. They show that this simple algorithm with $\delta = 0.01$ already gives results differing by no more than 0.004 from those with $\delta = 0.001$. These results all lie within the 95% confidence limits of the Monte Carlo estimates. The binomial/trinomial tree method also gives similar results, but its first entry in Table 1 falls outside the Monte Carlo confidence interval while its last entry in Table 1 is near the left end-point of the confidence interval. In Tables 3 and 4, the results given by the recursive integration algorithm with $\delta \leq 0.005$ are in close agreement with each other and with those obtained by Broadie *et al.* (1998) using the trinomial tree method. The results based on continuity correction are close to those of the recursive integration or trinomial tree method for $m \geq 80$, but do not have comparable accuracy for $m \leq 20$.

Because its complexity increases linearly with m , the recursive integration method becomes less attractive for large m . However, the number m of monitoring dates of a lookback option is typically less than 80, for which the recursive integration method is relatively fast. Moreover, for large values of m , one can switch to the continuity correction formulas of Broadie *et al.* (1998) which are

reasonably accurate for $m \geq 80$, as demonstrated by Tables 3 and 4. Therefore, we recommend a combined recursive integration/continuity correction approach in practice, using the former for $m \leq 80$ and switching to the latter for larger values of m .

From Relations 25 and 26, it follows that $\int_{A_n}^{\infty} e^x P\{U_n \in dx\} = o(\delta)$ if we choose $B \sim 2|\log \delta|^{1/2}$. With this choice of B and noting that $(\mu + \tilde{\sigma}^2/2)n \leq rT$ and $\tilde{\sigma}\sqrt{n} \leq \sigma\sqrt{T}$, it follows from Expressions 27 and 28, that, for fixed m , the computational complexity of our recursive integration method is $O(\delta^{-1}|\log \delta|^{1/2})$ to give results accurate to within $O(\delta)$. No comparable accuracy results have been established for the binomial/trinomial tree method, which is based on weak convergence of the tree to geometric Brownian motion as n , the number of steps, approaches ∞ . The best one can expect is that the results are accurate to within $O(n^{-1})$, but the convergence rate results for Donsker's invariance principle established by Sawyer (1968) and others only yield the order $O(n^{-1/2}(\log n)^\varepsilon)$ for every $\varepsilon > 0$. As noted by Kat (1995), the number of calculations required by a standard binomial tree with n steps is of the order $n^2/2$, which is a lower bound for the complexity of the Babbs/Cheuk–Vorst method. Hence if one wants the binomial/trinomial tree method to yield results accurate to within $O(\delta)$, then the number of time steps needed is at least some constant times δ^{-1} , resulting in a complexity of at least some constant times δ^{-2} for the tree method, in contrast with the $O(\delta^{-1}|\log \delta|^{1/2})$ complexity for the recursive integration method that is accurate to within $O(\delta)$. Although Breen (1991) has made use of Richardson extrapolation to develop an accelerated binomial tree method for standard options such that the number of calculations increases linearly with n , the convergence rate of this method may be slow and its extension to lookback options is unresolved.

5. Conclusion

In this paper we make use of the duality theory of random walks to derive simple recursive formulas for the valuation of several types of lookback options, in which monitoring occurs at equally spaced times. We also provide an efficient algorithm to evaluate the univariate integrals in these formulas. Beginning with the seminal work of Armitage *et al.* (1969), recursive integration has been used successfully in group sequential testing in clinical trials, where a trial is terminated early if some test statistic crosses a prespecified boundary at the time of an interim data analysis. This recursive integration approach was recently extended to the pricing of barrier options in AitSahlia and Lai (1997), and the present paper further extends the methodology to the pricing of lookback options.

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