Pricing and Hedging of American Knock-In Options

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American barrier options of the knock-in type involve non-Markovian optimal stopping problems for early exercise. They therefore cannot be priced via standard methods such as binomial or trinomial trees and finite-difference schemes for free-boundary partial differential equations.

This article provides a modified tree method to price these options. It also develops fast and accurate analytic approximations for the price and hedge parameters.

Complex derivatives have become accepted instruments to tailor risk coverage for risk managers and investors. Barrier-type options have become important instruments, particularly for the valuation of structured products (see Banks [1994]). They are also widely used in currency markets.

The holder of a barrier option acquires option coverage on only a subset of the risky outcomes for which a plain vanilla option pays off; this reduces the cost of the resulting coverage so that the holder of the contract does not have to pay for contingencies the holder thinks are unlikely to occur. Because of this flexibility, barrier options were traded over the counter long before the opening of the Chicago Board Options Exchange, and have become some of the most commonly traded derivative contracts. American barrier options offer the added flexibility of early exercise but have to be priced using numerical algorithms, as they do not have closed-form solutions, unlike their European-style counterparts (see Merton [1973] and Rubinstein and Reiner [1991]).

Naive application of the Cox-Ross-Rubinstein binomial tree method for barrier options has been shown by Boyle and Lau [1994] to yield inaccurate values, even with many steps. To address this problem, which stems from the position of the barrier relative to the grid, a number of variants of the tree method have been advanced.


The sum of American knock-in and knock-out prices does not equal the standard American option price, as is the case for European barrier options. Moreover, the knock-in option value process is non-Markovian, so the classic binomial or trinomial tree methods or numerical partial differential equations for...
standard American options cannot be applied directly to price American knock-in options. We first explain why this is the case, and contrast American knock-in and knock-out options.

Then we develop a modified binomial tree method to price and hedge American down-and-in puts. An alternative method is based on the price decomposition approach for standard American options. We propose an efficient approximation to implement this approach that leads to analytic approximations that have penny accuracy. Numerical results for both approaches are provided, and we also address the evaluation of hedge parameters.

I. AMERICAN KNOCK-IN AND KNOCK-OUT OPTIONS

Consider an underlying asset whose price process $S_t$ follows a geometric Brownian motion with volatility $\sigma$ and that pays dividends at rate $q$ in a market environment where the riskless rate of return is $r$. Let $H$ be a barrier for either a knock-in or a knock-out option. Let $T$ be the expiration date for any option on the asset. Let $g(S, K)$ denote the option's payoff when exercised at asset price $S$. Then, $g(S, K) = (S - K)^+$ is the payoff of a call option with exercise price $K$, and $g(S, K) = (K - S)^+$ is the payoff on the corresponding put. Define:

$$T^{(D)}_{H} = \inf\{t \leq T : S_t < H\}$$
$$T^{(U)}_{H} = \inf\{t \leq T : S_t > H\}$$

That is, $T^{(D)}_{H}$ (or $T^{(U)}_{H}$) is the first time the price of the underlying asset falls below (or rises above) the barrier $H$. Then, for any stopping (early exercise) time $\tau \leq T$:

$$E_S [e^{-r \tau} g(S_\tau, K)] = E_S [e^{-r \tau} g(S_\tau, K) 1_{\{\tau \geq T^{(D)}_{H}\}}] + E_S [e^{-r \tau} g(S_\tau, K) 1_{\{\tau < T^{(D)}_{H}\}}]$$  (1)

where $E_S[X]$ denotes the expectation of a random variable $X$ conditional on the initial value $S_0 = S$. When $\tau = T$, Equation (1) expresses the well-known relation between the price of a standard European option, on the left, and the prices of corresponding European knock-in and knock-out options, respectively, on the right.

Let $T_{a,b}$ denote the class of stopping times taking values between $a$ and $b$ with $a < b$. From Karatzas [1988], the price of a standard American option is:

$$V(S) = \sup_{\tau \in T_{0,T}} E_S [e^{-r \tau} g(S_\tau, K)]$$  (2)

and the prices of the corresponding American knock-in and knock-out options are, respectively:

$$V_{IN}(S) = \sup_{\tau \in T_{0,T}} E_S [e^{-r \tau} g(S_\tau, K) 1_{\{\tau \geq T^{(D)}_{H}\}}]$$  (3)

$$V_{OUT}(S) = \sup_{\tau \in T_{0,T}} E_S [e^{-r \tau} g(S_\tau, K) 1_{\{\tau < T^{(D)}_{H}\}}]$$  (4)

Since the suprema in Equations (2)-(4) are attained at different stopping times, the price of a standard American option cannot be decomposed as in (1) into the sum of the corresponding American knock-in and knock-out options.

Since $\{S_t, t \geq 0\}$ and $\{S_{\min(t,t_{\tau}^{(H)})}, t \geq 0\}$ are Markov processes, the standard American option expressed in Equation (2) and the American knock-out option in Equation (4) are associated with Markovian optimal stopping problems. We can express Equation (4) as $V_{OUT}(S, 0)$, where:

$$V_{OUT}(S, 0) = \sup_{\tau \in T_{0,\tau}} E_S [e^{-r (\tau - 0)} g(S_\tau, K) 1_{\{\tau < T^{(D)}_{H}\}} | S_\tau = S]$$  (5)

The optimal stopping problem in Equation (3) associated with an American knock-in option, which becomes effective only after $T^{(D)}_{H}$, is non-Markovian. Because of this, we cannot apply standard algorithms such as finite-difference methods for free-boundary PDEs or binomial trees to compute $V_{IN}(S)$. Instead we use the representation:

$$V_{IN}(S) = \int_0^T e^{-r \tau} V(H, \tau) P \{\tau^{(H)}_{H} \in dt | S_0 = S\}$$  (6)

where $V(H, \tau)$ denotes the price of a standard American option (with maturity $T$ and strike $K$) at time $\tau$ when $S_\tau = H$.

II. MODIFIED BINOMIAL METHOD FOR AMERICAN KNOCK-IN PUTS

We can compute Equation (6) using a modified binomial tree method. Even faster analytic approximations decompose the price of a standard American option into the sum of the corresponding European option price and an early exercise premium.

To fix the ideas, we consider from now on the case
of down-and-in puts. Typically investors use put options as insurance against possible drops in the value of an asset they are holding. A down-and-in put enables its holder to reduce the cost of such insurance by requiring that it be effective only after the asset price falls below a barrier $H$.

We modify the binomial tree method to compute the price in Equation (6) of an American down-and-in put with $S > H$. While the usual binomial tree starts from the root node $S$ and may not include $H$ as a node value, our modified binomial tree uses a lattice that includes the barrier.

The integral in (6) involves the distribution of the first time that the geometric Brownian motion $S_t$ crosses the level $H$. Because $S_t = S \exp\{(r - q - \sigma^2/2)t + \sigma B_t\}$ under the risk-neutral measure $P$, where $\{B_t\}$ is a standard Brownian motion, and because the first-passage density of Brownian motion has a simple formula, we use the change of variables:

$$
z = \log S, \quad \gamma = \log H, \quad \lambda = r - q - \sigma^2/2$$

The knock-in time $\tau_{H}^{(0)} = \inf\{t \geq 0: S_t \leq H\}$ can then be expressed as $\tau_{H}^{(0)} = \inf\{t \geq 0: Z_t \leq \gamma\}$, where:

$$
z_t = z + \lambda t + \sigma B_t$$

is a Brownian motion with drift $\lambda$. Hence the probability distribution of $\tau_{H}^{(0)}$ has a density function $f_z$ given explicitly by:

$$f_z(t) = \left(\frac{z - \gamma}{\sigma \sqrt{2t}}\right) \times \left(\frac{\gamma - z - \lambda t}{\sigma \sqrt{t}}\right) \text{ for } t > 0$$

where $\phi(.)$ is the standard normal density function; see Karatzas and Shreve [1988, p. 196].

Using Equation (9), we can express the value (6) of an American down-and-in put (which we will denote $P_D(S)$ instead of $V_{(H,S)}$) as:

$$P_D(S) = \int_0^T e^{-rt} V(H, t) f_z(t) dt$$

$$\approx \delta \sum_{k=1}^M e^{-rt_k} V(e^{\gamma}, t_k) f_z(t_k)$$

Here the approximating Riemann sum involves $M + 1$ equally spaced time steps $t_0 = 0 < t_1 < \cdots < t_M = T$, with step size $\delta = T/M$ (so $t_k = k\delta$). To compute the $M$ American option prices $V(e^{\gamma}, t_{M}), V(e^{\gamma}, t_{M-1}), \ldots, V(e^{\gamma}, t_1)$ with a single run of the backward induction program, we approximate the Brownian motion $Z_t$ (with drift $\lambda$) by a Bernoulli random walk with time increment $\delta > 0$ and space increment $X$ such that

$$P\{X_i = \pm \sqrt{\delta(\sigma^2 + \lambda^2)^2}\} = \frac{1}{2}\left(1 \pm \frac{\lambda \sqrt{\delta}}{\sqrt{\sigma^2 + \lambda^2}}\right)$$

and is initialized at $T$ by $V(e^\gamma, T) = (K - e^\gamma)^+$. Note that $\log S$ may not belong to the lattice $L_\delta$, in contrast to the usual binomial tree method in which $S$ is always the root node of the tree but the barrier may not be a node of the tree. In AitSahlia, Imhof, and Lai [2003], we use a similar Bernoulli random walk with absorbing barrier $\gamma$ and increments (11) to approximate a Wiener process with the same absorbing barrier $\gamma$ to handle the barrier problem for knock-out options (see Boyle and Lau [1994]).

### III. FAST AND ACCURATE APPROXIMATION

Since the price $V(H, t)$ of a standard American put option can be decomposed as the sum of a European put plus an early exercise premium, we can likewise decompose the price (6) of an American down-and-in put as:

$$P_D(S) = p_D(S) + \int_0^T e^{-rt} \pi(H, T - t) f_z(t) dt$$

where $\pi(H, T - t)$ is the early exercise premium of a standard American put with maturity $T - t$, strike price $K$, and initial stock price $H$; $f_z(t)$ is given in (9); and $z = \log S$. The $p_D(S)$ in (13) is the price of a European down-and-in put option with strike price $K$ and expiration date $T$. In view of (1), $p_D(S)$ can be expressed as the difference between a standard European put and a European down-and-out put, yielding the closed-form expression:
\[ p_d(S) = -Se^{-qT}N(-d_1(S, H, T)) + Ke^{-rT}N(-d_2(S, H, T)) + Se^{-qT}(H/S)^{2+\epsilon}(N(d_1(H^2, SK, T)) - N(d_1(H, S, T))) - Ke^{-rT}(H/S)^{2}(N(d_2(H^2, SK, T)) - N(d_2(H, S, T))) \]

where \( d_1(x, y, \tau) = \frac{\log(x/y) + (r-q + \sigma^2/2)\tau}{\sigma\sqrt{\tau}} \)

\[ d_2(x, y, \tau) = d_1(x, y, \tau) - \sigma/\sqrt{\tau}, \text{ and } N(x) \text{ is the standard normal cumulative probability distribution function.} \]

At this point, the early exercise premium \( \pi(H, T-t) \) is the only piece needed to fully determine the price of an American down-and-in put using Equation (13).

Since this quantity is the difference between standard European and American option prices, for which various numerical methods are available, one may choose a method according to one's preference. Nevertheless, once \( \pi \) is determined, there remains the task of evaluating the integral in (13).

For fast and accurate approximations of the integrand and integral in (13), first we use Ju's [1998] or Ait-Sahalia and Lai's [2000] method to approximate the early exercise premium \( \pi \). The method involves approximating the early exercise boundary by a piecewise exponential function (Ju) in the original geometric Brownian motion scale or by a piecewise linear function (Ait-Sahalia and Lai) in the Brownian motion scale resulting from a change of variables. A major advantage of this approach is that it leads to closed-form approximations of the early exercise premium.

Multiplying the discounted American option premium \( e^{-r(t)}\pi(H, T-t) \) by the first-passage density \( f_\tau(t) \) then gives the value of the integrand in (13). For fast valuation, the integral has to be approximated by a sum of \( m \) terms with small \( m \). We evaluate it using Gaussian quadrature with \( m \) nodes and weight function \( f_\tau(t) \), which results in a weighted sum that approximates the integral in (13) (see Press et al. [1992, Section 4.5]).

The weights \( w_1, \ldots, w_m \) are determined together with nodes \( t_1, \ldots, t_m \) so that

\[ \int_0^T b(t)f_\tau(t)\,dt = \sum_{k=1}^m w_k b(t_k) \]

for all polynomials \( b \) of degree less than \( 2m \).

To find these nodes and weights, a first step is to evaluate the moments

\[ c_k = \int_0^T t^k f_\tau(t)\,dt \quad \text{for } k = 0, \ldots, 2m-1 \]

Note that \( c_0 \) is exactly the probability (under the risk-neutral measure) that the barrier is hit (i.e., knock-in occurs) during the life of the option. Let \( a = z - y \) and \( b = -\lambda \). Recalling that \( n(x) \) denotes the standard normal density and \( N(x) \) its cumulative distribution function, we have:

\[ \frac{c_0}{2} + \frac{bc_1}{2a} = \int_0^T \left( \frac{at^{-3/2}}{2\sigma} + \frac{bt^{-1/2}}{2\sigma} \right) n\left( \frac{at^{-1/2}}{\sigma} - \frac{b}{\sigma} \right) \,dt \]

\[ = N_1 - N_2 \quad \text{(15)} \]

where

\[ N_1 = 1 \]

\[ N_2 = \frac{a}{\sigma\sqrt{T}} - \frac{b\sqrt{T}}{\sigma} \]

Similarly:

\[ \frac{c_0}{2} - \frac{bc_1}{2a} = \exp\left( \frac{2ab}{\sigma^2} \right) (M_1 - M_2) \quad \text{(16)} \]

where \( M_1 = 1 \) and \( M_2 = N(a/(\sigma\sqrt{T}) + b\sqrt{T}/\sigma) \). From (15) and (16):

\[ c_0 = N_1 - N_2 + \exp\left( \frac{2ab}{\sigma^2} \right) (M_1 - M_2) \quad \text{(17)} \]

and \( c_1 = (a/b)\{2(N_1 - N_2) - c_0\} \) if \( b \neq 0 \).

To compute the higher moments, note that by partial integration:

\[ c_k = \beta_k - \frac{a^2}{(2k-1)\sigma^2} c_{k-1} + \frac{b^2}{(2k-1)\sigma^2} c_{k+1} \]

where

\[ \beta_k = \frac{a}{\sigma(k - \frac{1}{2})} T^{k - \frac{1}{2}} n\left( \frac{a - bT}{\sigma\sqrt{T}} \right) \]

Therefore, in the case \( b \neq 0 \):

\[ c_k = \left( \frac{2k - 3}{a^2} \right) c_{k-1} - \beta_k + \frac{a^2}{b^2} c_{k-2} \]

If \( b = 0 \), then:
\[ c_k = \beta_k - \frac{a^2}{(2k - 1)\sigma^2} c_{k-1} \]

The moments \( c_0, \ldots, c_{2m-1} \) can therefore be evaluated recursively. Let

\[
q_v(t) = \det \begin{pmatrix}
  c_0 & c_1 & \cdots & c_{v-1} & 1 \\
  c_1 & c_2 & \cdots & c_v & t \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  c_v & c_{v+1} & \cdots & c_{2v-1} & t^v \\
\end{pmatrix}
\]

\[ D = \det \begin{pmatrix}
  c_0 & c_1 & \cdots & c_{m-1} \\
  c_1 & c_2 & \cdots & c_m \\
  \vdots & \vdots & \ddots & \vdots \\
  c_{m-1} & c_m & \cdots & c_{2m-2} \\
\end{pmatrix} \]

Note that the polynomials \( q_0(t), \ldots, q_m(t) \) are orthogonal with respect to \( f_v(t) dt \) in \([0, T]\). The nodes \( t_1, \ldots, t_m \) are the zeros of \( q_m(t) \), and the weights \( w_1, \ldots, w_m \) are given by

\[ w_k = \frac{D^2}{q_m(t_k)q_{m-1}(t_k)} \text{ for } k = 1, \ldots, m \]

Finally, to evaluate the integral in (13) we use the approximation:

\[
\int_0^T e^{-rt} \pi(H, T - t) f_z(t) dt \approx \sum_{k=1}^{\infty} w_k e^{-rt_k} \pi(H, T - t_k) \] (18)

choosing \( m \) to be small for fast computation.

IV. NUMERICAL ILLUSTRATION

Our numerical examples for the two methods consider both a short-maturity put option on a security paying no dividend (Exhibit 1) and a long-maturity put option on a security paying a dividend at a constant rate (Exhibit 2). The middle four columns are generated by the modified binomial algorithm in Equations (10) and (12), for values of \( N \) equal to 1,000, 5,000, 10,000 and 20,000. The expected convergence of the algorithm is clearly noticeable.

The last columns in the Exhibits present the integral approximation method with \( m = 2 \). In this case, the price of the American knock-in put is obtained via the approximation (18) of the integral in (13).

Observe that with \( N = 1,000 \), the modified binomial algorithm generally yields penny accuracy, and that the integral approximation with \( m = 2 \) nodes, which is over ten times faster, is even more accurate. The integral approximation involves \( q_0 \) given in (17), which is equal to the probability of ever hitting the barrier during the life of the option, also tabulated in Exhibits 1 and 2.

Note that one could improve the accuracy of the integral approximation by increasing the number \( m \) of nodes, thus resulting in higher-degree polynomials \( q_m(t) \).

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**EXHIBIT 1**

American Down-and-In Put Option Prices: Modified Binomial Algorithm and Integral Approximation (with \( m = 2 \))

<table>
<thead>
<tr>
<th>S</th>
<th>H</th>
<th>Prob. of Hitting Barrier</th>
<th>Modified Binomial</th>
<th>Integral Approx.</th>
</tr>
</thead>
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<tr>
<td></td>
<td></td>
<td></td>
<td>( N = 1000 )</td>
<td>( N = 5000 )</td>
</tr>
<tr>
<td>75</td>
<td>70</td>
<td>0.5821</td>
<td>17.3026</td>
<td>17.3007</td>
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<td>80</td>
<td>70</td>
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<td>14.24</td>
<td>58.51</td>
</tr>
</tbody>
</table>

\( r = 0.06, q = 0, \sigma = 0.2, T = 0.5, K = 100. \)
**V. AMERICAN KNOCK-IN HEDGE PARAMETERS**

From (6), it follows that the hedge parameter $\Delta$ is given by:

$$
\Delta = \frac{\partial P_D}{\partial S}(S) = \int_0^T e^{-r(T-t)}V(H,t) \frac{\partial f_z}{\partial S}(t)(t)dt
$$

(19)

Since $z = \log S$, the chain rule yields:

$$
\frac{\partial f_z}{\partial S}(t) = \frac{1}{S} \frac{\partial f_z}{\partial z}(t)
$$

Moreover, from (9), it follows that:

$$
\frac{\partial f_z}{\partial z}(t) = \frac{1}{\sigma \sqrt{t}} \left(1 + (z - \gamma) \left(\frac{\gamma - z - \lambda t}{\sigma^2 t}\right)\right) \eta \left(\frac{\gamma - z - \lambda t}{\sigma \sqrt{t}}\right)
$$

Therefore, as in (10), we can evaluate $\Delta$ via:

$$
\Delta = \frac{\partial P_D}{\partial S}(S) + \int_0^T e^{-r(T-t)}V(H,t) \frac{\partial f_z}{\partial S}(t)(t)dt
$$

(20)

Similar expressions can be obtained for the hedge parameters gamma and theta. Note that (20) shows that the Bernoulli walk algorithm to compute $\Delta$ does not involve numerical differentiation. Alternatively, one can use the decomposition formula (13) to compute the hedge parameters. For example:

$$
\Delta = \frac{\partial P_D}{\partial S}(S) + \int_0^T e^{-r(T-t)}\pi(H,t) \frac{\partial f_z}{\partial S}(t)(t)dt
$$

(21)

Here $\frac{\partial f_z}{\partial S}$ is given in closed form because of (14), and so is $\frac{\partial f_z}{\partial S}$. To evaluate the integral in (21), we can also use Gaussian quadrature.

**VI. SUMMARY**

We have considered American knock-in options, for which the integral defining the early exercise premium is very different from that of an American knock-out option. Despite the non-Markovian nature of the associated optimal stopping problem, we have been able to develop a modified binomial algorithm to price American knock-in options. We have also given an alternative
approach that makes use of the classic decomposition formula for a standard American option and computes the early exercise premium by Gaussian quadrature.

The methods presented for American down-and-in put options can be modified for up-and-in put options and for the corresponding call options.

REFERENCES


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