Fast and accurate valuation of American barrier options

Farid AitSahlia
DemandTec, 2 Circle Star Way, San Carlos, CA 94070, USA

Lorens Imhof
Department of Statistics, RWTH Aachen, D–52056 Aachen, Germany

Tze Leung Lai
Department of Statistics, Stanford University, Stanford, CA 94305, USA

This article develops two fast and accurate methods to compute boundaries, prices and hedge parameters of American barrier options. An extensive numerical study that compares these approximations with benchmark values and other methods shows that they are highly accurate and improve the currently available approximations in both speed and accuracy.

1 Introduction

The explosive growth in the use of derivatives by investors and institutions has fueled the need for fast and accurate valuation. In the past two decades substantial progress has been made in this direction for standard American options. Geske and Johnson (1984) characterized American options as compound European options and then used the Richardson extrapolation (with three or four points typically) to approximate the option price. MacMillan (1986) and Barone-Adesi and Whaley (1987) approximated the partial differential equation (PDE) for the difference between American and European option prices by an ordinary differential equation (ODE) and thereby derived an approximate valuation formula. Broadie and Detemple (1996) developed upper and lower bounds for an American option and used a convex combination of these bounds with empirically determined weights to approximate the option price. Carr (1998) discretized the time dimension of the PDE with a few points and used a randomization method to approximate the option price. Huang *et al* (1996), Ju (1998) and AitSahlia and Lai (1999) developed approximations to the early exercise boundary and used them to derive approximations to American option prices and hedge parameters.

We thank Bin Gao for providing us with the code to implement the procedure proposed in Gao *et al* (2000), and acknowledge stimulating comments from participants of the Annual Risk Derivatives Conferences in Paris (April, 2001) and Boston (June, 2001), and Taipei International Conference on Quantitative Finance (July, 2001), in which some of these results were presented. We also thank Nengjiu Ju for providing us with the code he wrote to implement his method.

In particular, Huang *et al* (1996) approximated the early exercise boundary by a step function while Ju used a piecewise exponential function to approximate the boundary. Ju (1998) reported numerical studies showing that his method with $n = 3$ pieces substantially improves earlier approximations in the literature in speed and accuracy. AitSahlia and Lai (1999) carried out extensive computations of early exercise boundaries for a wide range of maturities, interest rates, dividend rates, volatilities and strike prices via reparametrization to reduce American option valuation to a single optimal stopping problem for standard Brownian motion, indexed by one parameter in the absence of dividends and by two parameters otherwise. Their results show that the early exercise boundary is well approximated by a piecewise exponential boundary which uses a small number of pieces, and explain why Ju's method has superior performance over previous approximation approaches.

The past decade also witnessed important developments in the valuation of American barrier options. Barrier options are widely used by institutional investors, banks and corporations in their risk management, and American-style options give their holders the additional flexibility of early exercise. Boyle and Lau (1994) pointed out that naive application of the Cox–Ross–Rubinstein binomial tree method can result in significant errors even when a large number of time steps is used because the barrier typically lies between two adjacent layers of nodes in the lattice. They proposed to reduce the size of errors by refining the partition so that the resulting lattice has layers as close as possible to the barrier. Ritchken (1995) introduced a trinomial tree method, Cheuk and Vorst (1996) used a time-dependent shift of the trinomial tree, while Figlewski and Gao (1999) developed an "adaptive mesh model" to address difficulties with lattice methods for barrier options. By using piecewise constant or piecewise exponential functions to approximate the early exercise boundaries, Gao *et al* (2000) derived approximations to the values and hedge parameters for American barrier options, similar to those for standard options. They measured the accuracy of these approximations by the differences from corresponding values computed by Ritchken's method with a large number (between 10,027 and 21,385) of time steps. Because of the computational task involved, they conducted numerical studies on two relatively small samples of 48 contracts each. Moreover, in the absence of known convergence results for Ritchken's method, the adequacy of these "benchmark values" may be questionable.

In this article we use an alternative approach to compute more efficiently benchmark values that are accurate up to an $O(n^{-1})$ error, where n is the number of time steps. It is based on a modification of the corrected Bernoulli walk method of Chernoff and Petkau (1986) to solve a canonical form of optimal stopping problems for American barrier options. This modification, given in Section 2, addresses the difficulties with lattice methods in the presence of a barrier and computes not only the option prices but also the entire exercise boundary. Using this approach, we carry out extensive computations, which show that the exercise boundaries of American barrier options are well approximated by piecewise exponential functions with a small number of pieces. Using this result, we develop in Sections 3 and 4 two fast and accurate approximations to the option prices and hedge parameters, which are compared in Section 5 with that of Gao *et al* (2000) and with other approximations in terms of speed and accuracy (using benchmark values computed by both our and Ritchken's methods that are in close agreement, but with ours much faster than Ritchken's method). In this connection, we also extend in Section 4 the approximation of MacMillan (1986) and Barone-Adesi and Whaley (1987) to American barrier options. Section 6 summarizes and concludes the article.

2 Benchmark values and exercise boundary

Under the usual assumptions of a riskless interest rate r and an underlying asset which follows a geometric Brownian motion (GBM) with volatility σ and which pays dividend at rate μ, the price of an American knock-out option at time $t \leq T$ (= expiration date) that entails no arbitrage opportunities before exercise is given by

$$P(t,S) = \sup_{\tau \in \mathcal{T}_{t,T}} E\left\{ e^{-r(\tau-t)} g(S_\tau) 1_{\{\tau<\tau_H\}} \middle| S_t = S \right\} \tag{1}$$

where $S_t = S_0 e^{(r-\mu-\sigma^2/2)t+\sigma W_t}$, with initial price S_0, $\{W_t\}$ is standard Brownian motion, τ_H is the first time that S_t crosses the barrier (to be defined precisely below), and $\mathcal{T}_{a,b}$ is the set of stopping times taking values between a and b with $b>a$; $g(S) = (S-K)^+$ for a call, or $(K-S)^+$ for a put. Here $\tau_H = \inf\{t \leq T\colon S_t \geq H\}$ for an "up-and-out" barrier, and $\tau_H = \inf\{t \leq T\colon S_t \leq H\}$ for a "down-and-out" barrier.

It is widely recognized that the usual binomial tree method to compute $P(0,S)$ has difficulties because the barrier H typically lies between two adjacent layers of nodes in the tree. To circumvent these difficulties, we fix H as a node in the tree at the expense of giving up the initial stock price S as the root node. In fact, following the previous work for standard American options in AitSahlia and Lai (1999), a Bernoulli random walk with absorbing barrier $\gamma = \ln(H/K)$ can be used to approximate a Wiener process with the same absorbing barrier γ. Specifically, using the change of variables

$$\rho = \frac{r}{\sigma^2}, \quad \alpha = \frac{\mu}{r}, \quad \lambda = \rho - \alpha\rho - \frac{1}{2} \tag{2}$$

$$\gamma = \ln\left(\frac{H}{K}\right), \quad z = \ln\left(\frac{S}{K}\right), \quad u = \sigma^2(t-T) \tag{3}$$

they expressed (1) as $Ke^{\rho u} w(u,z)$, where

$$w(u,z) = \sup_{\tau \in \mathcal{T}_{u,0}} E\left\{ e^{-\rho\tau} (1-e^{Z_\tau})^+ 1_{\{\tau<\tau(\gamma)\}} \middle| Z_u = z \right\} \tag{4}$$

in the case of puts, with $(1 - e^{Z_\tau})^+$ replaced by $(e^{Z_\tau} - 1)^+$ for calls, $Z_s = \lambda s + W_s$, $s \le 0$, is a Wiener process with drift λ and $\tau \equiv \tau(\gamma) = \inf\{s \le 0: Z_s \ge \gamma\}$. The approximating Bernoulli random walk has absorbing barrier γ, time increment $\delta > 0$ and space increment X_i such that

$$P\left\{X_i = \pm\sqrt{\delta}(1+\delta\lambda^2)^{\frac{1}{2}}\right\} = \frac{1}{2}\left(1 \pm \frac{\lambda\sqrt{\delta}}{\sqrt{1+\delta\lambda^2}}\right) \tag{5}$$

Note that X_i has mean $\lambda\delta = E(Z_{t+\delta} - Z_t)$ and variance $\delta = \mathrm{var}(Z_{t+\delta} - Z_t)$.

We shall focus on up-and-out puts with $H > K$ in the sequel. Let $L_\delta = \{\gamma - \sqrt{\delta}(1+\delta\lambda^2)^{1/2} j : j = 0, 1, 2, \dots\}$, $u_0 = 0$ and $u_k = u_{k-1} - \delta$ for $k \ge 1$. Approximate (4) by the backward recursion

$$w(u_{n+1}, z) = \max\left\{e^{-\rho u_{n+1}}(1-e^z)^+,\ Ew(u_n, z+X_n)\right\} \tag{6}$$

with boundary condition $w(u_i, \gamma) = 0$ for all i. Since (6) only computes $w(u_i, z)$ for $z \in L_\delta$, the value of $w(u_i, z)$ at $z \notin L_\delta$ can be computed by interpolation, eg, by using linear interpolation, or Lagrange's interpolation formula with a quadratic interpolation polynomial (see Press *et al*, 1992). Similarly the value of $w(u, z)$ for $u \notin \{u_0, u_1, \dots\}$ can also be obtained by interpolation. The stopping boundary of the discrete-time optimal stopping problem for the approximating Bernoulli walk is given by

$$\bar{z}_\delta(u_i) = \max\left\{z \in L_\delta : w(u_i, z) = e^{-\rho u_i}(1-e^z)^+\right\} \tag{7}$$

In addition to the Bernoulli walk approximation, we also use the following decomposition formula for $P(t, S)$ $(= e^{\rho u} w(u,z)$ in the case $K = 1)$:

$$\begin{aligned} e^{\rho u} w(u,z) &= f(u,z) - e^{2\lambda(\gamma-z)} f(u, 2\gamma - z) \\ &+ \rho \int_u^0 e^{-\rho(s-u)} \Big\{ N(d(\lambda, s, u, z)) - e^{2\lambda(\gamma-z)} N(d(\lambda, s, u, 2\gamma - z)) \\ &- \alpha e^z \Big[N(d(\lambda+1, s, u, z)) - e^{(2\lambda+2)(\gamma-z)} N(d(\lambda+1, s, u, 2\gamma - z)) \Big] \Big\} \mathrm{d}s \end{aligned} \tag{8}$$

where $d(\xi, s, u, y) = \{\bar{z}(s) - y - \xi(s-u)\}/\sqrt{s-u}$ and

$$f(u,z) = e^{\rho u} N\left(-\frac{z - \lambda u}{\sqrt{-u}}\right) - e^{z + \alpha\rho u} N\left(-\frac{z-(\lambda+1)u}{\sqrt{-u}}\right)$$

(see Gao *et al*, 2000), with $\bar{z}(s)$ being the optimal stopping boundary for the continuous-time problem (4). In the transformed coordinates (3), $f(u,z)$ and $f(u,z) - e^{2\lambda(\gamma-z)} f(u, 2\gamma - z)$ are, respectively, the European standard and barrier option prices. Let $\psi(u,z)$ denote the RHS of (8). Setting $z = \bar{z}(u)$ in (8) yields the following integral equation for the exercise boundary:

$$1-\mathrm{e}^{\bar{z}(u)}=\psi(u,\bar{z}(u)) \tag{9}$$

Instead of initializing the recursion (6) at $n=0$ with $w(u_0,z)=(1-\mathrm{e}^z)^+$, we propose to initialize (6) at n_0 and determine $w(u_{n0},z)$ via (8), in which the integral can be expressed in closed form when $\bar{z}(u)$ for $u_i \le u \le u_{i-1}$ is obtained by linearly interpolating $\bar{z}(u_i)$ and $\bar{z}(u_{i-1})$. Using this closed form to compute the integral in (8), $\bar{z}(u_i)$ can be determined recursively from (9) for $0 \le i \le n_0$, with $\bar{z}(0)=-(\log\alpha)^+$. The details are given in Section 3. For $i>n_0$, $\bar{z}(u_i)$ is obtained from the Bernoulli optimal stopping boundary (7) via the Chernoff–Petkau (1986) correction as follows:

$$z_\delta^0(u_i)=\bar{z}_\delta(u_i)+\sqrt{\delta},\quad z_\delta^1(u_i)=\bar{z}_\delta(u_i)+2\sqrt{\delta},$$

$$D_j(u_i)=\mathrm{e}^{-\rho u_i}\left\{1-\exp\left(z_\delta^j(u_i)\right)\right\}^+-w\left(u_i,z_\delta^j(u_i)\right)\quad\text{for } j=0,1,$$

$$\bar{z}(u_i)=z_\delta^0(u_i)+\sqrt{\delta}\left|D_1(u_i)\ \ \{2D_1(u_i)-4D_0(u_i)\}\right|$$

This "hybrid" method thus combines the decomposition approach for u near 0 with the Bernoulli walk method for $u \le u_{n0}$. Theoretical analysis similar to that used in AitSahlia and Lai (2001) shows that the hybrid method yields option values with $O(\delta)$ error and the optimal stopping boundary $\bar{z}(\cdot)$ with $o(\sqrt{\delta})$ error.

Differentiating the decomposition (8) for $P(t,S)$ $(=\mathrm{e}^{\rho u}w(u,z))$ with respect to S $(=\mathrm{e}^z)$, we can express the hedge ratio in the canonical scale (3) as

$$\begin{aligned}\frac{\partial P}{\partial S}=&-\mathrm{e}^{\alpha\rho u}N\left(-\frac{z-(\lambda+1)u}{\sqrt{-u}}\right)-\mathrm{e}^{(2\lambda+2)(\gamma-z)}N\left(-\frac{2\gamma-z-(\lambda+1)u}{\sqrt{-u}}\right)\\
&+2\lambda\mathrm{e}^{2\lambda\gamma-(2\lambda+1)z}f(u,2\gamma-z)+\rho\int_u^0\mathrm{e}^{-\rho(s-u)}\Big\{\mathrm{e}^{-z}\Big[-\phi(\lambda,s,u,z)\\
&-\mathrm{e}^{2\lambda(\gamma-z)}\Big(\phi(\lambda+1,s,u,2\gamma-z)-2\lambda N\big(d(\lambda+1,s,u,2\gamma-z)\big)\Big)\Big]\\
&-\alpha\Big[\Big(1-\mathrm{e}^{(2\lambda+2)(\gamma-z)}N\big(d(\lambda+1,s,u,2\gamma-z)\big)-\big(1+\mathrm{e}^{(2\lambda+2)(\gamma-z)}\big)\Big)\\
&\times\Big(\phi(\lambda+1,s,u,2\gamma-z)-(2\lambda+2)N(d(\lambda+1,s,u,2\gamma-z))\Big)\Big]\Big\}\mathrm{d}s\end{aligned} \tag{10}$$

where $\phi(\xi,s,u,y)=n(d(\xi,s,u,y))/\sqrt{s-u}$, with $n(\cdot)$ being the standard normal density function. Since $\bar{z}(\cdot)$ can be computed with $o(\sqrt{\delta})$ error, we can also compute (10) with $o(\sqrt{\delta})$ error by using the closed-form integration formula in Section 3 that linearly interpolates between $\bar{z}(u_i)$ and $\bar{z}(u_{i-1})$. Other hedge parameters such as gamma, vega and rho can be computed similarly.

3 An efficient approximation

In this section we first consider the computation of the early exercise premium and solution of the integral equation (9). Our numerical results show that $\bar{z}(\cdot)$ is well approximated by a linear spline with a few knots. We then make use of such approximations to develop a fast method to compute option values, hedge parameters and $\bar{z}(\cdot)$ approximately.

To evaluate the integral in (8), we use Ju's (1998) closed-form expression for the integral when $\bar{z}(\cdot)$ is piecewise linear. Note that in view of the transformation (3), a piecewise linear $\bar{z}(\cdot)$ corresponds to the piecewise exponential B_t used by Ju to approximate the exercise boundaries of standard American options. Specifically, assuming that $u = u_m < \ldots < u_0 = 0$, partition $[u, 0]$ into m subintervals (not necessarily of equal width) such that

$$\bar{z}(s) = b_i s + \alpha_i \quad \text{for } u_i \le s \le u_{i-1} (1 \le i \le m) \tag{11}$$

and letting

$$\tau_i = u_i - u, \; a_i = \sqrt{b_i^2 + 2\rho}, \; c_i = \alpha_i + b_i u - z + \lambda u$$

Ju showed that for $1 \le i \le m$,

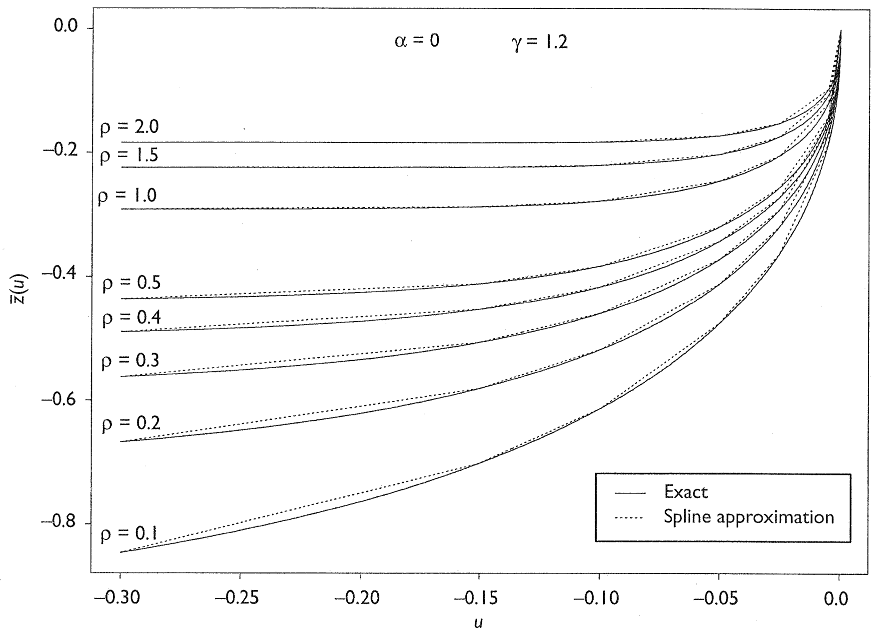
$$\begin{aligned}
\int_{\tau_i}^{\tau_{i-1}} \rho e^{-\rho t} N\left(\frac{\bar{z}(t+u) - z + \lambda u}{\sqrt{t}}\right) dt &= \int_{\tau_i}^{\tau_{i-1}} \rho e^{-\rho t} N\left(b_i \sqrt{t} + \frac{c_i}{\sqrt{t}}\right) dt \\
&= e^{-\rho \tau_i} N\left(b_i \tau_i^{1/2} + c_i \tau_i^{-1/2}\right) - e^{-\rho \tau_{i-1}} N\left(b_i \tau_{i-1}^{1/2} + c_i \tau_{i-1}^{-1/2}\right) \\
&+ \frac{1}{2}\left(\frac{b_i}{a_i} + 1\right) e^{(a_i - b_i) c_i} \left\{ N\left(a_i \tau_{i-1}^{1/2} + c_i \tau_{i-1}^{-1/2}\right) - N\left(a_i \tau_i^{1/2} + c_i \tau_i^{-1/2}\right) \right\} \\
&+ \frac{1}{2}\left(\frac{b_i}{a_i} - 1\right) e^{-(a_i + b_i) c_i} \left\{ N\left(a_i \tau_{i-1}^{1/2} - c_i \tau_{i-1}^{-1/2}\right) - N\left(a_i \tau_i^{1/2} - c_i \tau_i^{-1/2}\right) \right\}
\end{aligned} \tag{12}$$

In the case $\tau_i = 0$, we replace τ_i by $t > 0$ and take the limit as $t \to 0$, which amounts to setting $c_i \tau_i^{-1/2} = 0$, or ∞, or $-\infty$ according as $c_i = 0$, or $c_i > 0$, or $c_i < 0$. The other integrands in (8) can be treated similarly.

To solve the integral equation (9) numerically, let $\bar{z}_j = \bar{z}(u_j)$. Suppose $\bar{z}_1, \ldots,$ $\bar{z}_{m-1}$ have already been determined. Let z be a candidate value for $\bar{z}_m$ and let $b(z) = (z - \bar{z}_{m-1})/(u_m - u_{m-1})$, $\alpha(z) = z - b(z)u_m$. Linearly interpolating $\bar{z}(\cdot)$ between u_j and u_{j-1} for $1 \le j \le m$ gives the piecewise linear function (11) with $b_m = b(z)$ and $\alpha_m = \alpha(z)$. In view of (9), using the preceding closed-form expression to evaluate the integral in (8) leads to the following nonlinear equation for $\bar{z}_m (= z)$:

$$1 - e^z = \psi(u_m, z) \tag{13}$$

FIGURE 1 Optimal stopping boundaries: up-and-out put.

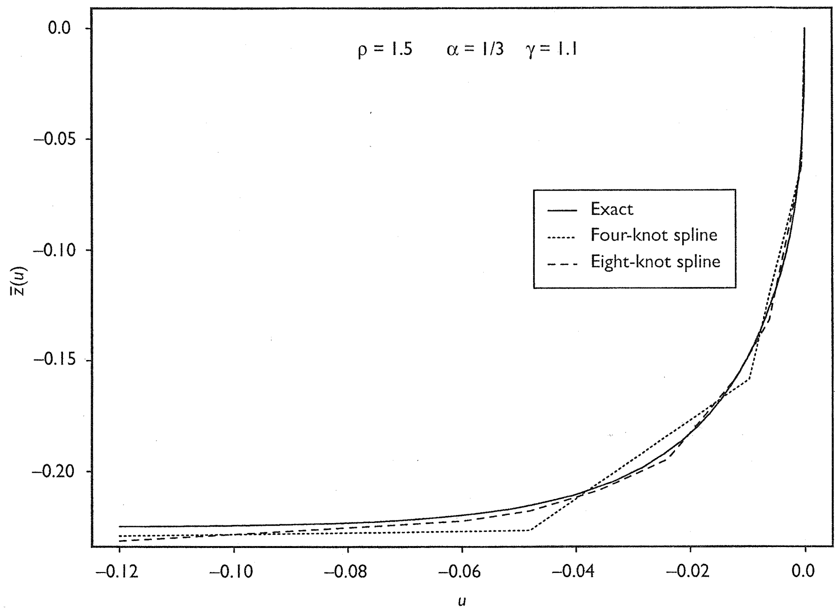


which can be solved by a Van Wijngaarden–Dekker–Brent-type method; see Press *et al* (1992).

This method to compute $\bar{z}(u_i)$ is fast when the number m of time steps is small. However, because of the iterative procedure to solve (13) and the non-recursive computations required to compute $\psi(u_m, z)$, it becomes computationally expensive for large m. This is the rationale behind the hybrid method in Section 2, which only uses (9) to compute the early exercise boundary for u near 0 (where $\bar{z}'(u)$ is large) and to initialize the recursive Bernoulli walk method at u_{n0}, where $u_{i-1} - u_i \equiv \delta$ with a small δ.

Figure 1 plots the graphs of $\bar{z}(u)$ for $-0.3 \leq u \leq 0$, $\gamma = 1.2$, $\alpha = 0$ and different values of ρ, computed by the hybrid method with $\delta = 10^{-4}$. It shows that $\bar{z}(\cdot)$ is well approximated in each case by a linear spline with six knots. The approximate piecewise linearity of $\bar{z}(\cdot)$ suggests that we can in fact use a small number n of time steps in the preceding procedure to solve (9) for $\bar{z}(\cdot)$. It often suffices to choose n as small as 4, with unevenly spaced knots at $-\sigma^2 T$, $(-0.4)\,\sigma^2 T$, $(-0.08)\,\sigma^2 T$ and $(-0.005)\,\sigma^2 T$. Figure 2 plots such an approximation to $\bar{z}(\cdot)$ in the case $\gamma = 1.1$, $\alpha = 1/3$, $\rho = 1.5$ and $\sigma^2 T = 0.12$, showing that the approximation is quite good. Doubling the number of time steps, with eight unevenly spaced knots at –1, –0.8, –0.6, –0.4, –0.2, –0.1, –0.05 and –0.005 multiplied by $\sigma^2 T$, gives a closer approximation.

FIGURE 2 Optimal stopping boundaries: up-and-out put.



Once the linear spline approximation to $\bar{z}(\cdot)$ is determined, we can use (8) and (12) to compute option values. The hedge ratio can be computed via (10), (12) and the following analogue of (12):

$$\int_{\tau_i}^{\tau_{i-1}} \frac{e^{-\rho t}}{\sqrt{t}} n\left(\frac{\bar{z}(t+u)-z+\lambda u}{\sqrt{t}}\right) dt = \int_{\tau_i}^{\tau_{i-1}} \frac{e^{-\rho t}}{\sqrt{t}} n\left(b_i\sqrt{t}+\frac{c_i}{\sqrt{t}}\right) dt$$

$$= (2\pi)^{-1/2} e^{-b_i c_i} \int_{\tau_i}^{\tau_{i-1}} t^{-1/2} e^{-(a_i^2 t + c_i^2 t^{-1})/2} dt$$

$$= a_i^{-1} e^{(a_i-b_i)c_i} \left\{N\left(a_i\tau_{i-1}^{1/2}+c_i\tau_{i-1}^{-1/2}\right) - N\left(a_i\tau_i^{1/2}+c_i\tau_i^{-1/2}\right)\right\}$$

$$+ a_i^{-1} e^{-(a_i+b_i)c_i} \left\{N\left(a_i\tau_{i-1}^{1/2}-c_i\tau_{i-1}^{-1/2}\right) - N\left(a_i\tau_i^{1/2}-c_i\tau_i^{-1/2}\right)\right\} \quad (14)$$

The last equality above can be derived by using the change of variables

$$x = a_i t^{1/2} + c_i t^{-1/2}, \quad y = a_i t^{1/2} - c_i t^{-1/2}$$

so that

$$dx = \tfrac{1}{2}\left(a_i t^{-1/2} - c_i t^{-3/2}\right) dt, \quad dy = \tfrac{1}{2}\left(a_i t^{-1/2} + c_i t^{-3/2}\right) dt$$

and by the identity

$$t^{-1/2} = \frac{\left(a_i t^{-1/2} - c_i t^{-3/2}\right) + \left(a_i t^{-1/2} + c_i t^{-3/2}\right)}{2a_i}$$

4 Extensions of MacMillan, Barone-Adesi and Whaley, and Ju

When there is no barrier, Ju (1998) used the following method to determine a piecewise exponential approximation to the original early exercise boundary B_t. The interval $[0, T]$ is divided into a small number (three or four) of evenly spaced pieces, and on the ith piece B_t is approximated by an exponential function $Q_i e^{q_i t}$. Although piecewise exponential B_t is equivalent to piecewise linear $\bar{z}(u)$, Ju's method is different from that in Section 3 because unlike the linear spline, the piecewise exponential approximation is not assumed to be continuous, resulting in $2n$ parameters $q_1^{(n)}, Q_1^{(n)}, \ldots, q_n^{(n)}, Q_n^{(n)}$ for an n-piece exponential approximation instead of the n parameters $\bar{z}(u_1), \ldots, \bar{z}(u_n)$ for the n-piece linear spline. The $(q_m^{(n)}, Q_m^{(n)})$ are determined recursively by solving two nonlinear equations $f_1(q_m^{(n)}, Q_m^{(n)}) = 0$ and $f_2(q_m^{(n)}, Q_m^{(n)}) = -1$, where f_1 comes directly from the integral equation (9) which can be expressed in the original variables as

$$\begin{aligned} K - B_t = p(t, B_t) + \int_t^T e^{-r(\tau - t)} \Big\{ & rK\Big[N(-d_2(B_t, B_\tau, \tau - t)) \\ & - (H/B_t)^{2\lambda} N\left(-d_2\left(H^2/B_t, B_\tau, \tau - t\right)\right)\Big] \\ & - \mu B_t\Big[N(-d_1(B_t, B_\tau, \tau - t)) \\ & - (H/B_t)^{2\lambda+2} N\left(-d_1\left(H^2/B_t, B_\tau, \tau - t\right)\right)\Big]\Big\} d\tau \end{aligned} \qquad (15)$$

in which $p(t, S)$ is the European barrier put price and

$$\begin{aligned} d_1(x, y, \tau) &= \frac{\ln(x/y) + \left(r - \mu + \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}} \\ d_2(x, y, \tau) &= d_1(x, y, \tau) - \sigma\sqrt{\tau} \end{aligned} \qquad (16)$$

Differentiating both sides of (15) with respect to the critical price B_t yields the equation $f_2 = -1$.

Good starting values are needed to solve the nonlinear simultaneous equations $f_1 = 0, f_2 = -1$ for $(q_m^{(n)}, Q_m^{(n)}))$ by Newton's method. When there is no barrier, Ju (1998) uses $(q_1^{(j-1)}, Q_1^{(j-1)})$ to initialize $(q_1^{(j)}), Q_1^{(j)})$ for $2 \le j \le n$ and $(q_{i-1}^{(n)}, Q_{i-1}^{(n)})$ to initialize $(q_i^{(n)}, Q_i^{(n)})$ for $2 \le i \le n$. To initialize $(q_1^{(1)}, Q_1^{(1)})$ for the one-piece approximation, he uses (0, B^*) with B^* given by the approximation to B_0 due to MacMillan (1986) and Barone-Adesi and Whaley (1987). To extend his approach to American barrier options, we shall develop the corresponding approximation B^* when there is a barrier. After computing such B^*, we can also follow his procedure of replacing the n-piece exponential function by a step function (ie, $q_i^{(n)} = 0$ for all i) if $|B^* - B_T|/B_T \le 0.05$ (with $B_T = \min\{Kr/\mu, K\}$) to avoid nonconvergence of Newton's method when the n-piece exponential function is relatively flat.

4.1 Extension of MacMillan, Barone-Adesi and Whaley

Let $p(t,S)$ and $P(t,S)$ denote, respectively, the price of a European and an American barrier put when the underlying security has price S at time t. Define ρ and α by (2) and let $\rho^* = 2\rho$, $\alpha^* = 2\rho(1-\alpha)$, $\tau = T - t$. In the continuation region, the early exercise premium $\pi(\tau, S)$, defined as $P(T-\tau,S) - p(T-\tau,S)$, satisfies the Black–Scholes PDE

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \pi}{\partial S^2} + (r-\mu) S \frac{\partial \pi}{\partial S} - \frac{\partial \pi}{\partial \tau} = r\pi$$

Following MacMillan (1986) and Barone-Adesi and Whaley (1987), we first seek solutions of the form

$$\pi(\tau,S) = g(\tau) h(g(\tau),S) \tag{17}$$

where $g(\tau) = 1 - e^{-r\tau}$. In other words, h must satisfy the PDE

$$S^2 \frac{\partial^2 h}{\partial S^2} + \alpha^* S \frac{\partial h}{\partial S} - \frac{\rho^*}{g} h - (1-g)\rho^* \frac{\partial h}{\partial g} = 0 \tag{18}$$

We next drop the term $(1-g)\rho^* \,\partial h/\partial g$ in (18) to obtain the ODE

$$S^2 \frac{\partial^2 h}{\partial S^2} + \alpha^* S \frac{\partial h}{\partial S} - \frac{\rho^*}{g} h = 0$$

a general solution of which is

$$h(\tau,S) = a_1(\tau) S^{b_1(\tau)} + a_2(\tau) S^{b_2(\tau)} \tag{19}$$

where

$$b_1(\tau) = \frac{-(\alpha^*-1) - \sqrt{(\alpha^*-1)^2 + 4\rho^*/g(\tau)}}{2}$$

$$b_2(\tau) = \frac{-(\alpha^*-1) + \sqrt{(\alpha^*-1)^2 + 4\rho^*/g(\tau)}}{2}$$

To fully specify h in (17) we rely on the barrier and optimal exercise conditions to determine $a_1(\tau)$ and $a_2(\tau)$. Clearly $h(\tau, H) = 0$ and thus $a_2(\tau) = -a_1(\tau)H^{b_1(\tau) - b_2(\tau)}$. Let $\bar{S}$ be the critical price at $t = T - \tau$. Then the optimal exercise conditions at $(t, \bar{S})$ can be expressed as

$$K - \bar{S} = p(T - \tau, \bar{S}) + a_1 g\left[\bar{S}^{b_1} - H^{b_1}\left(\bar{S}/H\right)^{b_2}\right], \tag{20}$$

$$-1 = \frac{\partial p}{\partial S}(T - \tau, \bar{S}) + a_1 g\left[b_1 \bar{S}^{b_1 - 1} - b_2 H^{b_1 - 1}\left(\bar{S}/H\right)^{b_2 - 1}\right] \tag{21}$$

where to simplify notation we omit τ from $a_1(\tau)$, $a_2(\tau)$, $b_1(\tau)$, $b_2(\tau)$ and $g(\tau)$. From (21) we obtain

$$a_1 = -\frac{1 + (\partial p/\partial S)(T - \tau, \bar{S})}{g\left[b_1 \bar{S}^{b_1 - 1} - b_2 H^{b_1 - 1}\left(\bar{S}/H\right)^{b_2 - 1}\right]} \tag{22}$$

Combining (22) with (20) yields

$$K - \bar{S} = p(T - \tau, \bar{S}) - \left\{1 + \frac{\partial p}{\partial S}(T - \tau, \bar{S})\right\} \frac{\bar{S}^{b_1} - H^{b_1}\left(\bar{S}/H\right)^{b_2}}{b_1 \bar{S}^{b_1 - 1} - b_2 H^{b_1 - 1}\left(\bar{S}/H\right)^{b_2 - 1}} \tag{23}$$

where $(\partial p/\partial S)(T - \tau, \bar{S})$ is the hedge ratio for the corresponding European barrier put at $t = T - \tau$ and $S = \bar{S}$, while $p(T - \tau, \bar{S})$ is given by $f(u, z) - e^{2\lambda(\gamma - z)} f(u, 2\gamma - z)$ in (8) after the change of variables (3). We can solve the nonlinear equation (23) for $\bar{S}$ by a Van Wijngaarden–Dekker–Brent-type method. In particular, $\bar{S}$ determined in this way at $t = 0$ (or equivalently $\tau = T$) gives the value of B^* for initializing the one-piece exponential approximation to the exercise boundary in Ju's approach.

4.2 Implementation issues

As indicated above, Ju's approach requires solving two simultaneous nonlinear equations for the parameters q_i, Q_i of each of the n exponential functions $Q_i e^{q_i t}$ that are pieced together to approximate the exercise boundary. These equations are solved by Newton's method that uses an elaborate initializing scheme, beginning with a one-piece, followed by a two- (and then three-, etc.) piece approximation. For standard American options, Ju recommends choosing $n = 3$ and his computer program that implements this approach only uses three iterations in Newton's method to solve for each (q_i, Q_i) plus six iterations to solve for the B^* that initializes the one-piece approximation. This ensures a fast procedure. Moreover, since the one- and two-piece approximations have to be computed before the three-piece approximation, it takes little extra work to compute the option price (or hedge ratio) by using a three-point Richardson extrapolation scheme to combine the prices (or hedge ratios) computed from the one-, two- and three-piece approximations to the exercise boundary.

The same implementation can be applied to American knock-out options, to which we have extended the method of MacMillan, Barone-Adesi and Whaley. However, there are many more terms to compute, especially in the dividend-paying case. Solving two simultaneous equations $f_1(q_i^{(n)}, Q_i^{(n)}) = 0$, $f_2(q_i^{(n)}, Q_i^{(n)}) = -1$ by Newton's method requires computation of many terms that appear in the first and second partial derivatives of the right hand side of (15) with respect to the critical price, and this makes Ju's method for up-and-out puts substantially slower in the presence than in the absence of dividends.

In their extension of Ju's approximation to barrier options, Gao *et al* (2000) bypass the preceding extension of MacMillan, Barone-Adesi and Whaley to initialize the one-piece exponential approximation. Instead of following Ju to initialize the solution for $(q_1^{(n)}, Q_1^{(n)})$, they initialize at $B_T(= K \min\{1, r/\mu\})$. Their procedure, which we denote by G^{EXP}, will be compared in the next section with the preceding extension of Ju's procedure.

5 Numerical results

In this section we demonstrate, through a large sample study, the accuracy and speed of the spline approximation to $\bar{z}(u)$ in Section 3 based only on a few knots, and of our extension of Ju's method in Section 4 using a three-piece exponential approximation to B_t. We also include in the study our extension of MacMillan, Barone-Adesi and Whaley, which is used to initialize Ju's method. Also included in the study is G^{EXP}, which is the procedure of Gao *et al* (2000) to approximate B_t with a three-piece exponential function and which we implemented by using a computer program supplied by Gao. Besides the piecewise exponential approximation, Gao *et al* (2000) have also proposed to approximate B_t by a piecewise constant function (step function), which we denote by G^{STEP} in our numerical study. These approximations are compared with benchmark values computed by using the hybrid method in Section 2 with $\delta = 10^{-4}$. Also included for comparison with benchmark values is Ritchken's (1995) method with 10,000 and 800 time steps. All computations were performed on a shared Silicon Graphics Challenge workstation with four 64-bit Mips processors.

Table 1 reports the results from 1200 American up-and-out options where the parameters are independently drawn from the following distributions: time to maturity T is uniform between $1/24$ and three years, the barrier H has probability $1/3$ of assuming each value in $\{110, 150, 200\}$, the current stock price S is uniform between $0.9 \times H$ and 80, the volatility σ is uniformly distributed on [0.1, 0.6] and the riskless interest rate r is uniformly distributed on [0.02, 0.15]. In addition, $K = 100$ and $\mu = 0$. The prices are in dollars and the entries of Table 1 represent the differences in cents between the prices generated by the hybrid method of Section 2 (benchmark) and the alternative methods.

In Table 1, the columns labeled R10000 and R800 represent Ritchken's method with the number of time steps fixed at 10,000 and 800, respectively. The columns labeled SP4 and SP8 correspond to the linear spline approximation

TABLE 1 Summary of price deviation (in cents) from benchmark values (defined in Section 2) for 1,200 randomly generated puts in no-dividend case

	R10000	R800	SP4	SP8	MBW	Ju	G^{STEP}	G^{EXP}
Max abs. error (cents)	0.46	1.15	4.84	2.21	27.63	0.81	6.38	6.25*
Max rel. error (%)	0.12	0.28	2.65	0.71	15.21	0.59	0.43	0.53*
RMSE (cents)	0.07	0.13	0.66	0.18	6.18	0.16	1.56	0.48*
RMSRE (%)	0.01	0.03	0.13	0.07	1.99	0.04	0.13	0.05*
# abs. err. > 1 cent	0	1	143	6	882	0	409	21*
# rel. err. > 1%	0	0	2	0	249	0	0	0*
CPU time (sec)	15799	118	1.81	6.81	0.68	7.30	1.60	37.67

Starting with R10000, the columns represent, respectively, from left to right the method of Ritchken with 10,000 steps and 800 steps, spline approximations with four and eight pieces, our extensions of MBW (MacMillan, Barone-Adesi and Whaley) and of Ju (three-piece exponential approximation), and the piecewise constant and exponential approximations of Gao *et al*. The asterisk indicates that the entries apply to only 849 out of the original 1200 puts; see text. The CPU time for generating the benchmark option prices by the hybrid method was 504.5 seconds, much faster than R10000.

in Section 3, where $\bar{z}(\cdot)$ is approximated by, respectively, four- and eight-point splines at $-1, -0.4, -0.08, -0.005$ ($\times\, \sigma^2 T$), and at $-1, -0.8, -0.6, -0.4, -0.2, -0.1, -0.05, -0.005$ ($\times\, \sigma^2 T$). The column labeled Ju corresponds to our extension of Ju's method in Section 4 using a three-piece approximation to B_t. The column labeled MBW corresponds to our extension of the method of MacMillan (1986) and Barone-Adesi and Whaley (1987) in Section 4.1. The columns labeled G^{EXP} and G^{STEP} correspond to the methods of Gao *et al* (2000) that approximate B_t with a three-piece exponential and with a step function, respectively. Table 1 shows that both SP8 and Ju have the best performance in terms of accuracy and speed, and Ju has smaller maximum error. SP4 and G^{STEP} are faster but less accurate procedures, with SP4 having a smaller root mean squared error (RMSE) than G^{STEP}. The G^{EXP} method only converged in 851 out of the 1,200 cases; the entries marked by asterisks (*) in Table 1 are actually for 849 of these 851 contracts. The inclusion of the remaining two contracts for which there was convergence would have raised the maximum absolute error and the RMSE considerably (to 378 and 13 cents, and to 102 and 3.5 cents if only the largest is excluded).

Although both Ju and G^{EXP} are based on piecewise exponential approximations to B_t with three evenly spaced pieces, they differ in their choice of starting values for solving the simultaneous nonlinear equations defining the parameters of the exponential functions, as explained in Section 4.2. It is well known that two-dimensional root finding algorithms are sensitive to starting values; see Press *et al* (1992). Somehow the meticulous initialization scheme used in Ju has resulted in a numerically stable solution, while the simpler and more direct initialization scheme used in G^{EXP} has resulted in nonconvergence for about 30% of the cases in Table 1. In fact Newton's procedure to solve the simultaneous

TABLE 2 Option prices for American up-and-out puts ($r = 0.04$, $\sigma = 0.2$, $T = 3$, $K = 100, \mu = 0$)

S	H	Benchmark	R10000	R800	SP4	SP8	MBW	Ju	G^{STEP}	G^{EXP}
80	101	20.0000	20.0000	20.0000	20.0000	20.0000	20.0000	20.0000	20.0000	20.0000
85	101	15.0000	15.0000	15.0000	15.0000	15.0000	15.0000	15.0000	15.0000	15.0000
90	101	10.0000	10.0000	10.0000	10.0000	10.0000	10.0000	10.0000	10.0000	10.0000
95	101	5.1604	5.1608	5.1600	5.1615	5.1608	5.1495	5.1608	5.1608	5.1608
100	101	0.8166	0.8161	NA	0.8163	0.8161	0.8133	0.8161	0.8161	0.8161
80	105	20.0000	20.0000	20.0000	20.0000	20.0000	20.0000	20.0000	20.0000	20.0000
85	105	15.0384	15.0392	15.0371	15.0366	15.0421	15.0280	15.0391	15.0401	15.0392
90	105	10.5791	10.5793	10.5793	10.5731	10.5854	10.5551	10.5787	10.5813	10.5788
95	105	6.6511	6.6512	6.6497	6.6457	6.6565	6.6316	6.6506	6.6531	6.6506
100	105	3.1516	3.1513	3.1511	3.1483	3.1542	3.1413	3.1509	3.1524	3.1510
80	110	20.0128	20.0137	20.0139	20.0133	20.0158	20.0066	20.0139	20.0160	20.0138
85	110	15.5123	15.5131	15.5129	15.5104	15.5199	15.4875	15.5129	15.5221	15.5129
90	110	11.6162	11.6161	11.6155	11.6128	11.6232	11.5948	11.6157	11.6271	11.6157
95	110	8.2000	8.1998	8.1993	8.1969	8.2054	8.1870	8.1995	8.2098	8.1995
100	110	5.1709	5.1706	5.1705	5.1686	5.1743	5.1644	5.1703	5.1778	5.1704
105	110	2.4563	2.4562	2.4560	2.4552	2.4579	2.4539	2.4560	2.4598	2.4561
80	120	20.3520	20.3514	20.3508	20.3638	20.3598	20.2912	20.3524	20.3669	20.3524
85	120	16.4154	16.4146	16.4141	16.4294	16.4257	16.3445	16.4166	16.4400	16.4166
90	120	13.0507	13.0504	13.0507	13.0634	13.0605	12.9945	13.0527	13.0778	13.0526
95	120	10.1485	10.1480	10.1483	10.1582	10.1561	10.1108	10.1500	10.1722	10.1499
100	120	7.6192	7.6190	7.6192	7.6264	7.6250	7.5975	7.6204	7.6378	7.6204
105	120	5.3926	5.3921	5.3924	5.3972	5.3963	5.3812	5.3930	5.4051	5.3929
110	120	3.4100	3.4096	3.4099	3.4127	3.4122	3.4048	3.4100	3.4175	3.4100
115	120	1.6251	1.6247	1.6248	1.6262	1.6259	1.6231	1.6249	1.6284	1.6249

Benchmark values are generated by the method described in Section 2 with $\delta = 10^{-4}$. Starting with R10000, the columns represent, respectively, from left to right the method of Ritchken with 10,000 steps and 800 steps, spline approximations with four and eight pieces, our extensions of MBW (MacMillan, Barone-Adesi and Whaley) and of Ju (three-piece exponential approximation), and the piecewise constant and exponential approximations of Gao *et al.* For the entry NA, see text.

equations was terminated in each of these cases because of a singular (or nearly singular) Jacobian matrix.

In situations where G^{EXP} gives numerically stable results, the prices computed by G^{EXP} and Ju are typically close to each other. This is illustrated on a set of 24 contracts in Table 2, where benchmark values computed by the hybrid method in Section 2 are also given, together with the option prices (in dollars) obtained by other methods. Ritchken's method with 800 time steps, R800, cannot be implemented when the current stock price S is too close to the barrier H (S = 100,

TABLE 3 Hedge ratios for American up-and-out puts ($r = 0.04$, $\sigma = 0.2$, $T = 3$, $K = 100, \mu = 0$)

S	H	Benchmark	R10000	R800	SP4	SP8	Ju	G^{STEP}	G^{EXP}
80	101	−1.0000	−1.0000	−1.0000	−1.0000	−1.0000	−1.0000	−1.0000	−1.0000
85	101	−1.0000	−1.0000	−1.0000	−1.0000	−1.0000	−1.0000	−1.0000	−1.0000
90	101	−1.0000	−1.0000	−1.0000	−1.0000	−1.0000	−1.0000	−1.0000	−1.0000
95	101	−0.9167	−0.9167	−0.9167	−0.9167	−0.9167	−0.9167	−0.9167	−0.9167
100	101	−0.8244	−0.8243	NA	−0.8245	−0.8244	−0.8244	−0.8243	−0.8243
80	105	−1.0000	−1.0000	−1.0000	−1.0000	−1.0000	−1.0000	−1.0000	−1.0000
85	105	−0.9541	−0.9536	−0.9537	−0.9559	−0.9529	−0.9538	−0.9533	−0.9538
90	105	−0.8354	−0.8349	−0.8350	−0.8361	−0.8353	−0.8349	−0.8348	−0.8349
95	105	−0.7403	−0.7398	−0.7397	−0.7406	−0.7408	−0.7398	−0.7399	−0.7398
100	105	−0.6633	−0.6628	−0.6629	−0.6634	−0.6640	−0.6627	−0.6630	−0.6627
80	110	−0.9711	−0.9706	−0.9708	−0.9717	−0.9692	−0.9707	−0.9687	−0.9707
85	110	−0.8357	−0.8351	−0.8352	−0.8361	−0.8352	−0.8351	−0.8342	−0.8351
90	110	−0.7284	−0.7278	−0.7278	−0.7288	−0.7286	−0.7278	−0.7277	−0.7278
95	110	−0.6425	−0.6418	−0.6419	−0.6429	−0.6429	−0.6418	−0.6423	−0.6418
100	110	−0.5729	−0.5723	−0.5723	−0.5733	−0.5734	−0.5722	−0.5729	−0.5722
105	110	−0.5161	−0.5154	−0.5155	−0.5165	−0.5165	−0.5154	−0.5161	−0.5154
80	120	−0.8537	−0.8534	−0.8536	−0.8522	−0.8522	−0.8532	−0.8506	−0.8532
85	120	−0.7264	−0.7260	−0.7260	−0.7265	−0.7262	−0.7258	−0.7248	−0.7258
90	120	−0.6240	−0.6234	−0.6235	−0.6247	−0.6242	−0.6234	−0.6236	−0.6234
95	120	−0.5412	−0.5405	−0.5406	−0.5421	−0.5415	−0.5406	−0.5414	−0.5406
100	120	−0.4742	−0.4735	−0.4735	−0.4752	−0.4745	−0.4736	−0.4746	−0.4736
105	120	−0.4200	−0.4192	−0.4193	−0.4210	−0.4203	−0.4193	−0.4203	−0.4193
110	120	−0.3761	−0.3753	−0.3754	−0.3771	−0.3764	−0.3754	−0.3763	−0.3754
115	120	−0.3406	−0.3398	−0.3399	−0.3415	−0.3409	−0.3399	−0.3406	−0.3399

Benchmark values are generated by the method described in Section 2 with $\delta = 10^{-4}$. Starting with R10000, the columns represent, respectively, from left to right the method of Ritchken with 10,000 steps and 800 steps, spline approximations with four and eight pieces, our extensions of Ju (three-piece exponential approximation), and the piecewise constant and exponential approximations of Gao *et al*. For the entry NA, see text.

$H = 101$ in Table 2); in this case a larger number of steps is needed. Note that R10000 is within 0.09 cents of the benchmark values for all 24 contracts, consistent with the results of the large sample study in Table 1.

Table 3 gives the values of the hedge ratios of these 24 contracts computed by different methods. The benchmark values are computed via (10), (12) and (14), in which the boundary $\bar{z}(\cdot)$ is determined by the hybrid method of Section 2 with $\delta = 10^{-4}$. The approximations SP4, SP8, Ju, G^{EXP} and G^{STEP} also compute the hedge ratios via (10), (12) and (14), but use linear spline approximations with four or eight knots to approximate $\bar{z}(\cdot)$ or piecewise exponential/constant functions to approximate B_t with three evenly spaced pieces. The tree methods R10000 and R800 use numerical differentiation to determine the hedge ratios.

6 Summary and discussion

By using the hybrid method of Section 2 and the reparametrization (2)–(3) to perform extensive computations of the exercise boundaries over a wide range of maturities, interest rates, dividend rates, volatilities and barrier/strike prices, we have found that the exercise boundaries of American knock-out options can be well approximated by continuous piecewise exponential functions that use a small number of pieces, or equivalently for the transformed coordinates (3), by linear splines with a few knots. Note in this connection that the time horizon $\sigma^2 T$ under the transformation (3) is only a small fraction of the maturity T. These findings have led us to two approximations of the option prices and hedge parameters, whose accuracy and speed relative to the benchmark values derived from the hybrid method and also to other competing methods have been assessed in a large sample study.

The first approximation, considered in Section 3, uses a linear spline with a few unevenly spaced knots to approximate the exercise boundary $\bar{z}(\cdot)$ in the coordinate system (3). The uneven spacing of the knots has the advantage of following the actual boundary more closely. A simple rule is devised for knot placement in the four-knot and eight-knot schemes of Sections 3 and 5. The slope of each linear piece of the spline can be found by solving a one-dimensional nonlinear equation, as the intercept is fixed by continuity of the spline. An important advantage of this piecewise linear approximation (or piecewise exponential approximation in the original coordinates), first noted by Ju (1998), is that it leads to a closed-form expression for the integral defining the early exercise premium; moreover, there are similar closed-form expressions for the hedge parameters, as shown in Section 3.

The second approximation, considered in Section 4, removes the continuity requirement in the piecewise exponential function that approximates B_t. This has the advantage that each piece is approximated separately, without relying on the previously fitted pieces. Of course the previously fitted pieces still have an impact on the accuracy of the current piece since they appear in the integral defining the early exercise premium, but they are not used directly as parameter(s) (such as the intercept of $\bar{z}(\cdot)$ in the spline approximation) of the current piece. It is, therefore, less important to choose the endpoints of each piece to match closely the actual boundary, and the simple choice of evenly spaced endpoints proposed by Ju (1998) has been found to perform well. The disadvantage of this flexibility is that we now have two (instead of one) parameters to determine for each piece of B_t, resulting in a system of two nonlinear equations that require good starting values, and our numerical study has demonstrated the sensitivity of two-dimensional root-finding algorithms to starting values. Ju

(1998) has developed an elaborate scheme to initialize Newton's method for solving these equations and has found that it works well in the case of standard American options. In the course of generalizing it to barrier options, we have also extended the classical method of MacMillan (1986) and Barone-Adesi and Whaley (1987). As shown in Table 1 and also noted in AitSahlia and Lai (2001) for standard American options, other equally plausible starting values to initialize Newton's method for solving the nonlinear equations in Ju's approximation can result in singular or ill-conditioned matrices in the Newton-type iterations.

REFERENCES

AitSahlia, F., and Lai, T.L. (1999). A canonical optimal stopping problem for American options and its numerical solution. *Journal of Computational Finance*, **3**, 33–52.

AitSahlia, F., and Lai, T.L. (2001). Exercise boundaries and efficient approximations to American option prices and hedge parameters. *Journal of Computational Finance*, **4**, 85–103.

Barone-Adesi, G., and Whaley, R. (1987). Efficient analytic approximations of American option values. *Journal of Finance*, **42**, 301–20.

Boyle, P., and Lau, S.H. (1994). Bumping up against the barrier with the binomial method. *Journal of Derivatives,* **2**, 6–14.

Broadie, M., and Detemple, J. (1996). American option valuation: New bounds, approximations and a comparison of existing methods. *Review of Financial Studies*, **9**, 1211–50.

Carr, P. (1998). Randomization and the American put. *Review of Financial Studies*, **11**, 597–626.

Chernoff, H., and Petkau, A.J. (1986). Numerical solutions for Bayes sequential decision problems. *SIAM Journal of Scientific and Statistical Computing*, **7**, 46–59.

Cheuk, T.H.F., and Vorst, T.C.F. (1996). Complex barrier options. *Journal of Derivatives*, **4**, 8–22.

Figlewski, S., and Gao, B. (1999). The adaptive mesh model: a new approach to efficient option pricing. *Journal of Financial Economics*, **53**, 313–51.

Gao, B., Huang, J., and Subrahmanyam, M.G. (2000). An analytical approach to the valuation of American path-dependent options. *Journal of Economic Dynamics and Control*, **24**, 1783–827.

Geske, R., and Johnson, H. (1984). The American put options valued analytically. *Journal of Finance*, **39**, 1511–24.

Huang, J., Subrahmanyam, M.G., and Yu, G.G. (1996). Pricing and hedging American options: a recursive integration method. *Review of Financial Studies*, **9**, 277–300.

Ju, N. (1998). Pricing an American option by approximating its early exercise boundary as a multipiece exponential function. *Review of Financial Studies*, **11**, 627–46.

MacMillan, L. (1986). Analytic approximation for the American put option. *Advances in Futures and Options Research*, **1**, 119–39.

Press, W., Teukolsky, S., Vetterling, W., and Flannery, B. (1992). *Numerical Recipes in C: The Art of Scientific Computing* (2nd edn), Cambridge University Press, New York.

Ritchken, P. (1995). On pricing barrier options. *Journal of Derivatives*, **3**, 19–28.