

The Lead-Lag Puzzle of Demand and Distribution: A Graphical Method Applied to Movies

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Appendix B: Analytic Definition and Application of Curvature to Lead-Lag Determination

Consider a parametric curve $\mathbf{X}(t) = (x(t), y(t))$, t defined on the real interval $[a, b]$. \mathbf{X} is a vector that traces the curve. Define the tangent vector $\mathbf{T}(t)$ as $\mathbf{X}'(t)$ and the curvature vector $\boldsymbol{\kappa}(t)$ as $\mathbf{T}'(t) = \mathbf{X}''(t) = (d^2x/dt^2, d^2y/dt^2)$. Note that \mathbf{T} is orthogonal to $\boldsymbol{\kappa}$, which points towards the center of curvature. Figure A5 illustrates this vector system.

==== Place Figures A5 about here =====

Now define a moving coordinate system with unit orthogonal vectors $\mathbf{e}_1, \mathbf{e}_2$ with respect to the curve. The system origin is at \mathbf{X} and $\mathbf{e}_1 = \mathbf{T} / |\mathbf{T}|$. Define \mathbf{e}_2 such that the orientation is the same as the fixed system x, y . That is, if x, y is a right-handed coordinate system, so is $\mathbf{e}_1, \mathbf{e}_2$. We can specify \mathbf{e}_2 as $J\mathbf{e}_1$ where the operator J is a rotation by $\pi/2$, or similarly, if $\mathbf{e}_1 = (v_1, v_2)$, then $\mathbf{e}_2 = (-v_2, v_1)$. When the parameterization of the curve is with respect to the arc length s (i.e., $t = s$), this system is known as a Frenet frame¹. For our purposes the parameter t need not be arc length.²

Define the *signed* curvature as the scalar $\kappa(t)$ s.t. $\boldsymbol{\kappa}(t) = \kappa(t)\mathbf{e}_2$. In a right-handed coordinate system as above, when $\kappa > 0$, the curvature is counterclockwise, and when $\kappa < 0$, the curvature is clockwise.

We next develop an analytic example following the examples in the text that relates the lag between two functions and the curvature of the associated state space plot. Consider a sinusoid defined on $[0, \pi]$, specifically $x(t) = a \cos(t)$, and a second function that is $x(t)$ lagged, specifically $y(t) = x(t - \pi/2)$. This example is chosen because it is easy to illustrate graphically, and to calculate the Frenet frame and curvature. In more complex cases the analysis can become intractable, and with non-analytic discrete time series the analytics cannot be done. In these cases the graphic interpretation of clockwise and counterclockwise can be much easier. Figure A6 illustrates the graphical patterns in the time domain and the corresponding state space.

¹ Talman, Richard (2000), *Geometric Mechanics*, New York, NY: John Wiley and Sons, pp 240-243.

² For an example of a moving Frenet frame, see <http://www.stedwards.edu/science/kelm/multi/frenet.html>.

To determine the sign of scalar $\kappa(t)$, we use the following deductions:

$$\mathbf{X}(t) = (a \cos(t), a \cos(t - \pi/2)) = (a \cos(t), a \sin(t)),$$

$$\mathbf{T}(t) = \mathbf{X}'(t) = (-a \sin(t), a \cos(t)),$$

$$\kappa(t) = \mathbf{T}'(t) = \mathbf{X}''(t) = (-a \cos(t), -a \sin(t)),$$

$$\mathbf{e}_1(t) = \mathbf{T} / |\mathbf{T}| = (-a \sin(t), a \cos(t)) / |\mathbf{T}|,$$

$$\mathbf{e}_2(t) = \mathbf{J}\mathbf{e}_1(t) = (-a \cos(t), -a \sin(t)) / |\mathbf{T}|,$$

$$\kappa(t) = 1/|\mathbf{T}| \mathbf{e}_2(t), \text{ and therefore } \kappa(t) > 0 \text{ for all } t \text{ in } [0, \pi]$$

In this example, the curvature of $\mathbf{X}(t)$ is analytically positive and graphically counterclockwise for all t . If one reverses the lag and makes y lead x , i.e., $y = x(t + \pi/2)$, and repeats the calculations, the result will be $\kappa(t) = -1/|\mathbf{T}| < 0$. The curvature is clockwise. The diagrams in Figure A7 show the curvature vector and the Frenet frames for the two cases.

==== Place Figures A6 and A7 about here =====

Figure A5. The Vector System Used to Define Trajectory Directions

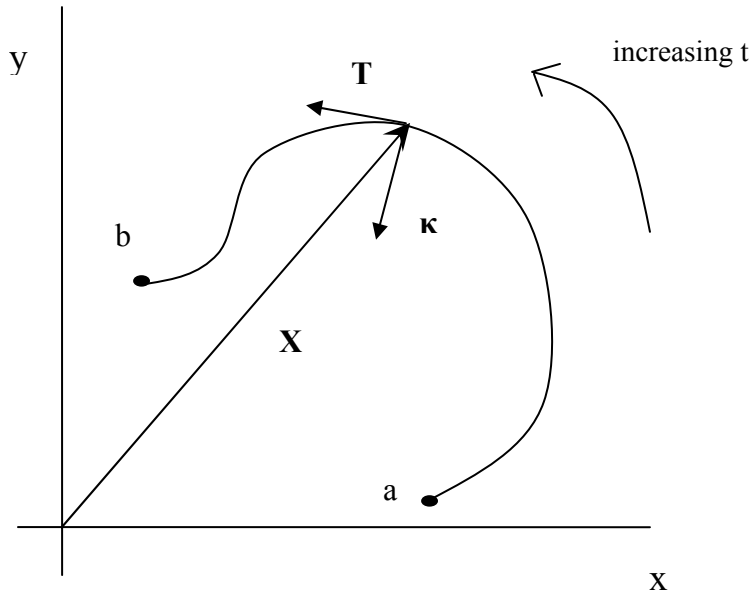


Figure A6. An Example of Lead-lag in the Time Domain and the State Space Domain

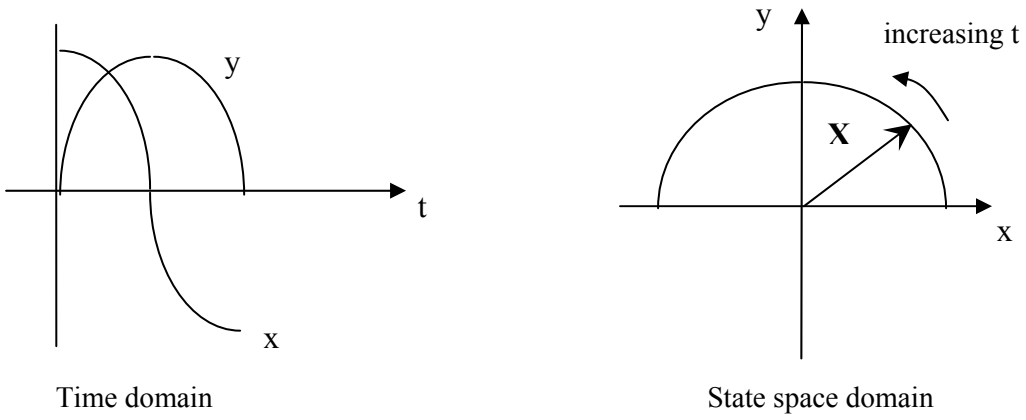
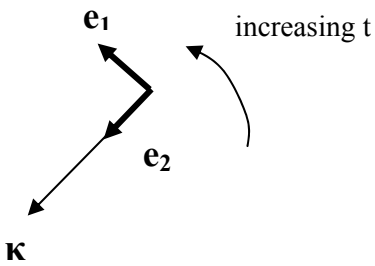
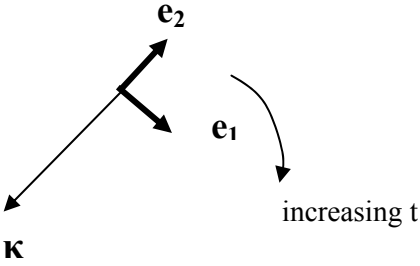


Figure A7. The Curvature Vector and the Frenet Frames for Different Curvatures



Positive curvature,
counterclockwise



Negative curvature,
clockwise